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Locally Non-spherical Artin Groups

Andrew Chermak

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506

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INTRODUCTION

Let S be a non-empty set and let M be an $S \times S$ matrix (i.e., a set indexed by $S \times S$) with entries in $\mathbb{N} \cup \{\infty\}$. We write $M = (m_{a,b})_{S \times S}$. We will say that M is a *Coxeter matrix* if M is symmetric, with $m_{a,b} \geq 2$ if and only if $a \neq b$.

Let $F = F(S)$ be the free group with free basis S. For any $m \in \mathbb{N}$ and any $a, b \in S$, let $w(a, b; m)$ denote the element of F which consists of the alternating product of the generators a and b, starting with a, and having length m. Put $w(a, b; \infty) = 1$, the identity element of F. If M is a Coxeter matrix over S then the *Artin group* $G = G(M)$ is the group presented by S subject to the relations

$$w(a, b; m_{a,b}) = w(b, a; m_{a,b})$$

for all $a, b \in S$. We say further that M and G are *locally non-spherical* if for each three-element subset $\{a, b, c\}$ of S we have

$$\frac{1}{m_{a,b}} + \frac{1}{m_{a,c}} + \frac{1}{m_{b,c}} \leq 1. \tag{0.1}$$

An equivalent way to formulate this condition is to say that for any three distinct elements a, b, c of S, the Coxeter group

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^{m_{a,b}} = (ac)^{m_{a,c}} = (bc)^{m_{b,c}} = 1 \rangle$$

is infinite.

Our aim in this paper is to prove the following two results.

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THEOREM 1. *Let $G(M)$ be a locally non-spherical Artin group, where M is a Coxeter matrix over a finite set S. Then $G(M)$ has a solvable Word Problem.*

THEOREM 2. *Let $G = G(M)$ be a locally non-spherical Artin group, M a Coxeter matrix over S. Let T be a non-empty subset of S, let M_T be the submatrix of M indexed by $T \times T$, and let G_T denote the subgroup of G generated by the canonical image of T in G. Then G_T is canonically isomorphic to $G(M_T)$.*

An algorithm $\mathscr{A}$ for the Word Problem in $G(M)$ is given explicitly in (6.5), below. It is closely related to Dehn's algorithm in that it never increases the length of a word.

The hypothesis that G be locally non-spherical generalizes the hypothesis, made in [2, 1] that G be of "large type," i.e., that $m_{a,b}$ be strictly greater than 2 for all $a, b \in S$ with $a \neq b$. Here we permit $m_{a,b} = 2$ (and we even also permit $m_{b,c} = 2$, with $c \neq a$, provided that $m_{a,c} = \infty$). On the other hand, our results are appreciably weaker than those of [2, 1]. In particular, our results do not suffice to establish solvability of the Conjugacy Problem, nor are we able to show that $G(M)$ is torsion-free.

The method employed here is an "arboreal small-cancellation theory" that was first introduced in [4] and which has undergone some improvement in [5]. This means that we study a set $\mathscr{H}$ of hyperbolic isometries of a tree X, with $\mathscr{H} = \mathscr{H}^{-1}$, and with $\mathscr{H} \trianglelefteq F$ for a suitable subgroup F of $\operatorname{Isom}(X)$. Here F will be the free group $F(S)$, but viewed as a free product of infinite cyclic groups

$$F = \mathop{*}_{a \in S} \langle a \rangle .$$

We then take X to be the "standard tree" for this free product (and which is quite different from the Cayley graph of F with respect to S). For any a, $b \in S$ with $a \neq b$, put

$$h_{a,b} = w(a, b; m_{a,b}) w(b, a; m_{a,b})^{-1}. \tag{0.2}$$

Let $N_{a,b}$ denote the normal closure of $h_{a,b}$ in the subgroup $\langle a, b \rangle$ of F generated by a and b. Then $\langle a, b \rangle / N_{a,b}$ is a two-generator Artin group, for which plenty of information is available in [2, 1], and more recently from [3]. Next, for any a, b as above, we set

$$\mathscr{H}_{a,b} = \bigcup \{ g^{-1} N_{a,b} g : g \in F \} - \{1\},$$

and then define

$$\mathscr{H} = \bigcup \{ \mathscr{H}_{a,b} : a, b \in S, a \neq b \}.$$

Finally, set

$$N = \langle \mathscr{H} \rangle, \qquad G = F/N. \tag{0.3}$$

Then G may be identified with the Artin group $G(M)$.

Now in fact $\mathscr{H}$ is a set of hyperbolic isometries of X. That is, for any $h \in \mathscr{H}$, h fixes no vertex and inverts no edge of X. For any such h we define

$$a(h) = \min\{d(x, x \cdot h)\}_{x \in X},$$
$$A(h) = \{x \in X \colon d(x, x \cdot h) = a(h)\}.$$

We call $a(h)$ the *amplitude* of h, or the *hyperbolic length* of h, and we call $A(h)$ the *axis* of h. The following two quoted results (which are in fact quite elementary) are fundamental to what follows.

0.4. PROPOSITION [7; 6, Proposition 24, p. 63]. *Let h be a hyperbolic isometry of a tree X. Then*:

(a) *$A(h)$ is a regular subtree of X with valency equal to* 2.

(b) *h induces a translation of $A(h)$ of amplitude $a(h)$.*

(c) *Every $\langle h \rangle$-invariant subtree of X contains $A(h)$.*

(d) *If a vertex x of X is at distance d from $A(h)$ then $d(x, xh) = a(h) +$ $2d$ (see Fig.* 1).

0.5. PROPOSITION [6, Proposition 25, p. 63]. *Let g be an isometry of a tree X. Then the following conditions are equivalent.*

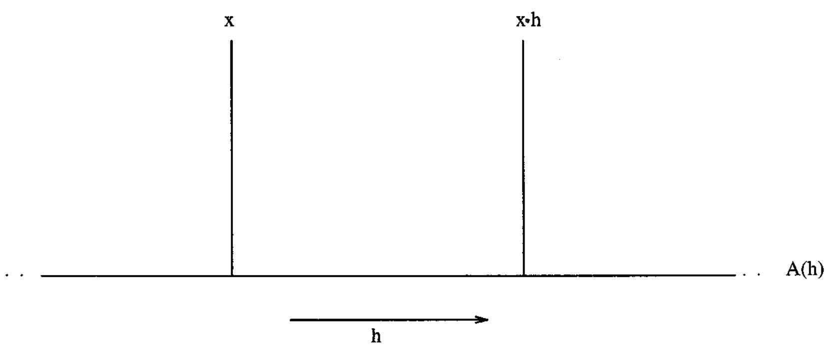
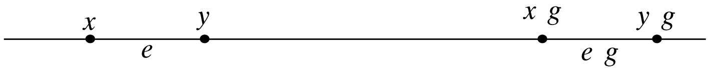


FIGURE 1

(i) *g is hyperbolic.*

(ii) *There is an edge e of X such that g translates e a positive distance along a geodesic path of X.*



Moreover, if (ii) *holds for the edge e then e is an edge of $A(g)$.*

The bulk of the present paper is devoted to showing that $\mathscr{H}$ has a certain kind of "convergence property" (as defined below, in (2.5)). For a discussion of "convergence" see the introduction to [4] and see especially [5]. This will occupy Sections 2 through 5. Section 6 is then a straightforward translation of this convergence property into Theorems 1 and 2.

Section 2 contains a few technical results concerning operations on finite sequences of isometries, as well as a review of some definitions from [5]. In particular, we remind the reader of the notion of a "polarization" of X over $\mathscr{H}$ in (2.6), and of what it means for a polarization $\mathscr{P}$ to "control convergence" in (2.8). After this, Section 3 is then possibly the most difficult to read, since it consists mostly of a chain of intricate definitions, culminating in the definition (3.12) of a suitable polarization $\mathscr{P}$. Sections 4 and 5 then perform the task of verifying that $\mathscr{P}$ controls convergence.

1. PRELIMINARIES

Fix a Coxeter matrix $M = (m_{a,b})_{S\times S}$, and let F denote the free group with free basis S. Define a graph X as follows. The vertex set $V(X)$ will be the (disjoint) union of two sets D and D' where D is simply the set F of group elements, and where D' is the set of all right cosets $\langle a\rangle g$ for $a \in S$ and $g \in F$. Then define an incidence relation between D and D' by means of membership: the element h of D and the element $\langle a\rangle g$ of D' are incident if $h \in \langle a\rangle g$, i.e., $h = a^n g$ for some n. Thus, X is a bipartite graph. In fact, Bass–Serre theory [6] shows that X is a tree, the "standard tree" for the free product F of the cyclic groups $\langle a\rangle$, $a \in S$.

Let d be the natural metric on $V(X)$. It will be convenient to identify X with its topological realization, so that X becomes a simplicial tree, with edges of length 1. Vertices of X will also be called *points*. For any points x and y in X, $[x, y]$ denotes the oriented closed path from x to y in X. We call this the *oriented closed segment* from x to y. If $J = [x, y]$ is an oriented closed segment we put

$$\partial_0 J = \{x\}, \qquad \partial_1 J = \{y\}, \qquad \partial J = \{x, y\}.$$

We may regard $[x, y]$ as a certain subset of X, together with a preferred direction. The underlying subset of $[x, y]$ is called the *closed segment* joining x and y, and it too will be denoted $[x, y]$. No confusion need result from this abuse of notation.

A closed segment is *degenerate* if it consists of a single point. We write $[x, y)$ for the subset $[x, y] - \{y\}$ of X, and we similarly define $(x, y]$ and (x, y). If $J = [x, y]$ is a closed segment, we write $l(J)$ for the *length* of J, which is $d(x, y)$.

Notice that F acts on X by right translation, and in this way every element of F may be viewed as an isometry of X.

1.1. LEMMA. (a) *Edge-stabilizers in F are trivial. Indeed, the stabilizer in F of any vertex in D is trivial.*

(b) *F is a subgroup of* $\mathrm{Isom}(X)$.

Proof. Part (a) is immediate, and (b) follows from (a).

Let $1 \neq g \in F$. We can then write

$$g = a_{i_1}^{e_1} \cdots a_{i_m}^{e_m},$$
$$\text{with } a_{i_j} \in S,\ e_j \in \mathbb{Z} - \{0\},\ \text{and with } a_{i_j} \neq a_{i_{j+1}} \text{ for any } j. \tag{1.2}$$

We call (1.2) the *normal form* of g, and we say that g is *cyclically reduced* if also $a_{i_m} \neq a_{i_1}$. Now set

$$|g| = |e_1| + \cdots + |e_m|, \qquad \text{and} \qquad \|g\| = m.$$

We refer to $|g|$ as the *length* of g, and to $\|g\|$ as the *syllable-length* of g.

1.3. LEMMA. *Let $g \in F$, with normal form as in* (1.2). *Then the vertex-sequence for the oriented closed segment* $[1, g]$ *is*

$$\left(1, \langle a_{i_m}\rangle, a_{i_m}^{e_m}, \langle a_{i_{m-1}}\rangle a_{i_m}^{e_m}, \ldots, \langle a_{i_1}\rangle a_{i_2}^{e_2} \cdots a_{i_m}^{e_m}, g\right).$$

In particular, we have $d(1, g) = 2 \cdot \|g\|$.

Proof. This is immediate.

For any $a, b \in S$ with $a \neq b$, we have defined (0.2) the basic relator:

$$h_{a,b} = w(a, b; m_{a,b})w(b, a; m_{a,b})^{-1}.$$

Denote by $N_{a,b}$ the smallest normal subgroup of $\langle a, b\rangle$ containing $h_{a,b}$. Thus, $\langle a, b\rangle / N_{a,b}$ is the Artin group with matrix

$$\begin{pmatrix} 1 & m_{a,b} \\ m_{a,b} & 1 \end{pmatrix}.$$

We require the following important result from [2].

1.4 [2, Lemma 7, p. 210]. *Let $g_1, g_2 \in \langle a, b\rangle$, and suppose that $g_1 g_2 \in N_{a,b}$.*

(a) *If $\|g_1\| \le m_{a,b}$ then $|g_1| \le |g_2|$.*

(b) *If $\|g_1\| < m_{a,b}$ then $|g_1| < |g_2|$ or $g_1 = g_2^{-1}$.*

1.5. COROLLARY (which is also [2, Lemma 6]). *For any $g \in N_{a,b}$, $g \ne 1$, we have $\|g\| \ge 2 \cdot m_{a,b}$.*

Let $X_{a,b}$ denote the subtree of X induced on the vertex set $\{g, \langle a\rangle g, \langle b\rangle g \colon g \in \langle a, b\rangle\}$. Certainly, $X_{a,b}$ is invariant under $\langle a, b\rangle$. Let h be a cyclically reduced element of $N_{a,b}$, $h \ne 1$. After possibly reversing the roles of a and b, we may then write $h = a^{e_1} b^{f_1} \cdots a^{e_l} b^{f_l}$, for some non-zero integers e_i and f_i, $1 \le i \le l$. By (1.3), the vertex-sequence for the oriented closed segment $[1, h]$ is then

$$(1, \langle b\rangle, b^{f_l}, \langle a\rangle b^{f_l}, \ldots, \langle a\rangle b^{f_1} \cdots a^{e_l} b^{f_l}, h),$$

and $d(1, h) = 2\|h\| = 4l$. Observe that $\langle a\rangle h = \langle a\rangle b^{f_1} \cdots a^{e_l} b^{f_l}$, so that the closed segment $[\langle a\rangle, 1]$ is translated a distance $2 \cdot \|h\|$ by h, along the segment $[1, h]$. Now (0.5) yields the following result.

1.6. LEMMA. *Let h be a cyclically reduced element of $N_{a,b}$, $h \ne 1$. Then h is a hyperbolic isometry of X, and $a(h) = 2 \cdot \|h\|$. Further, we have $A(h) \subseteq X_{a,b}$, and $A(h)$ contains the closed segment $[\langle a\rangle, \langle b\rangle]$ (whose vertex-sequence is $(\langle a\rangle, 1, \langle b\rangle)$).*

For any $a, b \in S$ with $a \ne b$, put

$$\mathscr{H}_{a,b} = \bigcup_{g \in F} \{g^{-1} N_{a,b} g\} - \{1\},$$

and then put

$$\mathscr{H} = \bigcup \{\mathscr{H}_{a,b} : a, b \in S, a \ne b\}.$$

Let $h \in \mathscr{H}$. Then h is conjugate to a cyclically reduced element of $N_{a,b}$, for some a and b, and hence h is a hyperbolic isometry of X, by (1.6). Now let g be a vertex in $D \cap A(h)$. The two neighbors of g in $A(h)$ are then vertices $\langle c\rangle g$ and $\langle d\rangle g$ for some $c, d \in S$ with $c \neq d$. Then $[\langle c\rangle, \langle d\rangle]$ is a segment of $A(h^{g^{-1}})$. On the other hand, we are given that there is a conjugate $h' = h^{g'}$ of h in $N_{a,b}$. Without loss, h' is cyclically reduced, and then $[\langle a\rangle, \langle b\rangle]$ is a segment of $A(h')$. Now gg' maps $A(h^{g^{-1}})$ onto $A(h')$, so gg' maps 1 into $X_{a,b}$. Thus, $gg' \in \langle a, b\rangle$, and so $\{c, d\} = \{a, b\}$. This yields parts (a) and (b) of the following result.

1.7. LEMMA. (a) *$\mathscr{H}$ is the disjoint union of the sets $\mathscr{H}_{a,b}$, $a, b \in S$, $a \neq b$.*

(b) *Let $h \in \mathscr{H}$. Then $h \in \mathscr{H}_{a,b}$ if and only if there exists $g \in F$ such that $[\langle a\rangle, \langle b\rangle] \cdot g$ is a closed segment of $A(h)$.*

(c) *Let $h \in \mathscr{H}_{a,b}$ and let $c \in S$. Suppose that $\langle c\rangle$ is a vertex of $A(h)$. Then $c \in \{a, b\}$ and $h \in N_{a,b}$.*

(d) *If $a, b \in S$ with $a \neq b$, and $[\langle a\rangle, \langle b\rangle]$ is a closed segment of $A(h)$, where $h \in \mathscr{H}$, then $h \in N_{a,b}$.*

Proof. We need only prove (c) and (d), and we note that (d) is an immediate consequence of (c). Now let $h \in \mathscr{H}_{a,b}$. It then follows from (a) and (b) that every vertex of $A(h)$ which is not in D is of the form $\langle a\rangle g$ or $\langle b\rangle g$ for some $g \in F$. If $c \in S$ and $\langle c\rangle$ is a vertex of $A(h)$, it now follows that $c \in \{a, b\}$. Without loss, $c = a$. Now $\langle a\rangle$ acts transitively on the vertices adjacent to the vertex $\langle a\rangle$, and since $N_{a,b}$ is $\langle a\rangle$-invariant we may then assume (after possibly replacing h by h^{a^n} for some n) that 1 is a vertex of $A(h)$. By assumption, $h^g \in N_{a,b}$ for some $g \in F$. Then g maps 1 into $X_{a,b}$, so $g \in \langle a, b\rangle$ and $h \in N_{a,b}$, proving (c).

We now come to two "Small Cancellation" results, which place restrictions on the ways in which axes for elements of $\mathscr{H}$ can intersect.

1.8. LEMMA. *Let $h, h' \in \mathscr{H}$ with $hh' \notin \mathscr{H} \cup \{1\}$. Then $l(A(h) \cap A(h')) \leq 2$, with equality only if $\partial(A(h) \cap A(h')) \subseteq D$.*

Proof. Suppose this is false. Then $A(h) \cap A(h')$ contains a closed segment I of length 2, with $\partial I \subseteq D'$. By (1.7)(b) we may replace h and h' by h^g and $(h')^g$ for a suitable $g \in F$, obtaining $I = [\langle a\rangle, \langle b\rangle]$ for some $a, b \in S$. But then $hh' \in N_{a,b}$ by (1.7)(d), and so $hh' \in \mathscr{H} \cup \{1\}$, for a contradiction.

In order to state the next result, we need to introduce the following notation. For any $h \in \mathscr{H}$, we put

$$m(h) = m_{a,b},$$

where $\{a, b\}$ is the uniquely determined subset of S such that $h \in \mathscr{H}_{a,b}$.

(1.9)

We will now make use of the hypothesis that our Artin group G is locally non-spherical.

1.10. LEMMA. *Let* h_1, h_2, h_3 *be three distinct elements of* $\mathscr{H}$. *Suppose that for all i and j with* $1 \leq i < j \leq 3$, *we have*

(i) $h_i h_j \notin \mathscr{H} \cup \{1\}$, *and*

(ii) $A(h_i) \cap A(h_j)$ *is a non-degenerate closed segment, oriented oppositely by* h_i *and* h_j.

Then $1/m(h_1) + 1/m(h_2) + 1/m(h_3) \leq 1$.

Proof. The conditions guarantee that there are closed segments $[x, y]$ of $A(h_1)$, $[y, z]$ of $A(h_2)$, and $[z, x]$ of $A(h_3)$, all of length 2, arranged as in Fig. 2.

Let w denote the vertex at the center of the picture. That is, $\{w\} = A(h_1) \cap A(h_2) \cap A(h_3)$. If we conjugate all three elements h_1, h_2, and h_3 by an element g of F, then the condition (i) remains invariant, as do the integers $m(h_i)$. We may therefore assume that either $w = 1$ or $w = \langle a \rangle$ for some $a \in S$.

Suppose $w = 1$. Then there are distinct elements a, b, c of S with $x = \langle a \rangle$, $y = \langle b \rangle$, and $z = \langle c \rangle$. By (1.7)(d) we get $h_1 \in N_{a,b}$, $h_2 \in N_{b,c}$, and $h_3 \in N_{a,c}$. As G is locally non-spherical, we have

$$\frac{1}{m_{a,b}} + \frac{1}{m_{a,c}} + \frac{1}{m_{b,c}} \leq 1$$

as desired.

Suppose $w = \langle a \rangle$ for some $a \in S$. By (1.7)(c) we then have $h_1 \in N_{a,b}$, $h_2 \in N_{a,c}$, and $h_3 \in N_{a,d}$, for some elements b, c, d of S. As $h_i h_j \notin \mathscr{H} \cup \{1\}$ for any $i \neq j$, it follows that b, c, and d are pairwise distinct. This then yields

$$\frac{1}{m_{a,b}} + \frac{1}{m_{a,c}} + \frac{1}{m_{a,d}} \leq 1,$$

as required.

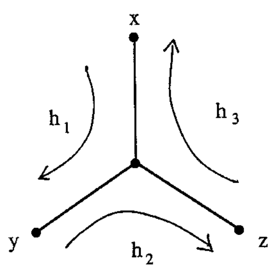


FIGURE 2

1.11. For any $h \in H$, put $b(h) = 4 \cdot m(h)$. Also, fix an isometry α_h from $\mathbb{Z}$ to $A(h)$, such that α_h carries the even integers onto $D \cap A(h)$. (Here we view $\mathbb{Z}$ as a simplicial tree, with edges $\{n, n+1\}$ of length 1.) The map $\mathscr{D}$ which assigns to each $h \in \mathscr{H}$ the triple $(\alpha_h, 2\mathbb{Z}, D \cap A(h))$ is then a *specialization* on $\mathscr{H}$, in the sense of [5, Definition (3.7)].

Observe that $b(h) \in 4\mathbb{Z}$, and $b(h) = b(h^{-1}) = b(h^g)$ for any $g \in F$. If we can show also that $b(h) \leq a(h)$, then $b(h)$ will be a *modified length function* on $\mathscr{H}$, in the sense of [5, Definition (3.8)]. In fact, such is the case, as we now prove.

1.12. LEMMA. *Let $h \in \mathscr{H}$. Then $a(h) = 2 \cdot \|h'\|$ for any cyclically reduced conjugate h' of h, and $a(h) \geq b(h)$. In particular, $a(h) \geq 8$.*

Proof. Since $a(h) = a(h')$ and $b(h) = b(h')$ for any conjugate h' of h, we may assume that h is cyclically reduced, and that $h \in N_{a,b}$ for some $a, b \in S$, $a \neq b$. Then $a(h) = 2\|h\|$ by (1.6), and so (1.5) yields $a(h) \geq 4 \cdot m_{a,b} = b(h)$.

2. $\mathscr{H}$-SEQUENCES

As in [5], an *$\mathscr{H}$-sequence* is always a finite sequence $\mathbf{h} = (h_0, \ldots, h_n)$ of elements of $\mathscr{H}$. We write $\mathbf{h}^{-1} = (h_n^{-1}, \ldots, h_0^{-1})$, and for any $g \in \langle \mathscr{H} \rangle$ we write $\mathbf{h}^g$ for $(h_0^g, \ldots, h_n^g)$. (Here x^g denotes $g^{-1}xg$ for any $x \in \mathscr{H}$.) We write $l(\mathbf{h})$ for the *length* of $\mathscr{H}$, which is here $n + 1$.

If $\mathbf{h}$ and $\mathbf{h}'$ are $\mathscr{H}$-sequences then $\mathbf{h} \circ \mathbf{h}'$ denotes the $\mathscr{H}$-sequence obtained by concatenating $\mathbf{h}$ and $\mathbf{h}'$. A *segment* of $\mathbf{h}$ is an $\mathscr{H}$-sequence $\mathbf{s}$ such that $\mathbf{h} = \mathbf{x} \circ \mathbf{s} \circ \mathbf{y}$ for some $\mathscr{H}$-sequences $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{x} = (\emptyset)$ we say that $\mathbf{s}$ is a *prefix* of $\mathbf{h}$, and we say that $\mathbf{s}$ is a *suffix* of $\mathbf{h}$ if $\mathbf{y} = (\emptyset)$.

The $\mathscr{H}$-sequence $\mathbf{h}$ is *reduced* if for all i, $1 \leq i \leq n$, we have $h_{i-1}h_i \notin \mathscr{H} \cup \{1\}$. We put $\mathscr{H}^{**} = \{\mathbf{h} : \mathbf{h}$ is a reduced $\mathscr{H}$-sequence$\}$. We recall the notion of braiding of $\mathscr{H}$-sequences, from Section 3 of [5]. Thus, the $\mathscr{H}$-sequence $\mathbf{h}' = (h'_0, \ldots, h'_n)$ is a *simple braiding* of $\mathbf{h}$ if there exists k with $1 \leq k \leq n$ such that $h'_i = h_i$ for all $i \notin \{k-1, k\}$, and either

$$(h'_{k-1}, h'_k) = \left(h_{k-1}h_k h_{k-1}^{-1}, h_{k-1}\right),$$

or

$$(h'_{k-1}, h'_k) = \left(h_k, h_k^{-1}h_{k-1}h_k\right).$$

If there exists a chain of $\mathscr{H}$-sequences

$$\mathbf{h} = \mathbf{s}_0, \ldots, \mathbf{s}_l = \mathbf{h}'$$

such that $\mathbf{s}_i$ is a simple braiding of $\mathbf{s}_{i-1}$ for all i, $1 \le i \le l$, then $\mathbf{h}'$ is a *braiding* of $\mathbf{h}$. In fact, there is an action of the Artin braid group B_{n+1} (on $n+1$ strings) on the set of $\mathscr{H}$-sequences of length $n+1$, such that the n standard generators of B_{n+1}, and their inverses, perform the simple braidings. We see then that $\mathbf{h}'$ is a braiding of $\mathbf{h}$ if and only if $\mathbf{h}'$ lies in the same orbit as $\mathbf{h}$ under the action of B_{n+1}. We put

$$\mathscr{H}^* = \{\mathbf{h} : \mathbf{h} \text{ is an } \mathscr{H}\text{-sequence, and every braiding of } \mathbf{h} \text{ is in } \mathscr{H}^{**}\}.$$

We quote the following result from [5, Lemma (3.6)].

2.0. LEMMA. (a) *Let* $\mathbf{h} = (h_0, \dots, h_n)$ *be a word of minimal length, in the alphabet* $\mathscr{H}$, *for the element* $h_0 \cdots h_n$ *of* $\mathrm{Isom}(X)$. *Then* $\mathbf{h} \in \mathscr{H}^*$.

(b) *Let* $\mathbf{h} = (h_0, \dots, h_n)$ *and* $\mathbf{h}' = (h'_0, \dots, h'_n)$ *be* $\mathscr{H}$*-sequences, and suppose that* h' *is a braiding of* $\mathbf{h}$. *Then* $h_0 \cdots h_n = h'_0 \cdots h'_n$.

2.1. DEFINITION. Let $\mathbf{h}$ and $\mathbf{h}'$ be $\mathscr{H}$-sequences, with $\mathbf{h} = (h_0, \dots, h_n)$, and let m be an integer, $0 \le m \le n$. We say that $\mathbf{h}'$ is a *simple distillation* of $\mathbf{h}$ (of *degree* m) if there exist k and l with $0 \le k \le l \le m$, such that

$$\mathbf{h}' = (h_0, \dots, h_{k-1})^{h_m} \circ (h_m) \circ (h_l, \dots, h_{m-1}) \circ (h_{m+1}, \dots, h_n).$$

In words: $\mathbf{h}'$ is obtained from $\mathbf{h}$ by permuting h_m to the left, across $(h_k, \dots, h_{m-1})$, deleting $(h_k, \dots, h_{l-1})$, and conjugating $(h_0, \dots, h_{k-1})$ by h_m. Notice that

$$l(h') = (n+1) - (l-k).$$

We will also say that the *codegree* of the above simple distillation is $l(\mathbf{h}) - m$.

EXAMPLE. Some simple distillations of (h_0, h_1, h_2, h_3) of degree 2 (and codegree 2) are (h_2, h_3), $(h_0^{h_2}, h_2, h_3)$, $(h_0^{h_2}, h_2, h_1, h_3)$, and (h_2, h_0, h_1, h_3).

2.2. DEFINITION. Let $\mathbf{h}$ and $\mathbf{h}'$ be $\mathscr{H}$-sequences, with $\mathbf{h} = (h_0, \dots, h_n)$. We say that $\mathbf{h}'$ is a *distillation* of $\mathbf{h}$ if $\mathbf{h} = \mathbf{h}' = (\emptyset)$, or if there exists a chain of $\mathscr{H}$-sequences

$$\mathbf{h} = \mathbf{s}_0, \dots, \mathbf{s}_n = \mathbf{h}' \qquad (n + 1 = l(\mathbf{h})),$$

such that for all m with $1 \le m \le n$, $\mathbf{s}_m$ is a simple distillation of $\mathbf{s}_{m-1}$ of codegree $n - m + 1$.

EXAMPLE. The sequence $(h_1^{h_2 h_3}, h_3)$ is a distillation of $(h_0, h_1, h_2, h_3) = \mathbf{s}_0$, as is revealed by taking $\mathbf{s}_1 = (h_1, h_2, h_3)$, $\mathbf{s}_2 = (h_1^{h_2}, h_2, h_3)$, and $\mathbf{s}_3 = (h_1^{h_2 h_3}, h_3)$. Here $n + 1 = 4$, and the reader may check, for example, that the codegree of the simple distillation from $\mathbf{s}_2$ to $\mathbf{s}_3$ is $1 = 4 - 3$.

2.3. LEMMA. *Let* $\mathbf{h} \in \mathscr{H}^*$, *let* $\mathbf{h}'$ *be a distillation of* $\mathbf{h}$, *and let* $h \in \mathscr{H}$ *such that* $\mathbf{h} \circ (h) \in \mathscr{H}^*$. *Then* $\mathbf{h}' \in \mathscr{H}^{**}$, *and for any entry* h' *of* $\mathbf{h}'$ *we have* $hh' \notin \mathscr{H} \cup \{1\}$ *and* $h'h \notin \mathscr{H} \cup \{1\}$.

Proof. By assumption there is a chain

$$\mathbf{h} = \mathbf{s}_0, \ldots, \mathbf{s}_n = \mathbf{h}'$$

such that each $\mathbf{s}_m$ is a simple distillation of $\mathbf{s}_{m-1}$ of codegree $n - m + 1$, where $n + 1 = l(\mathbf{h})$. Write $\mathbf{h} = (h_0, \ldots, h_n)$, as usual. Fix $m \geq 1$ and write

$$\mathbf{s}_{m-1} = (g_0, \ldots, g_r), \qquad \mathbf{s}_m = (g'_0, \ldots, g'_s),$$

where of course $n \geq r \geq s$. Put $m' = r + m - n$. We aim to prove the following statements, by induction on m.

(1)′ $\mathbf{s}_m = (g'_0, \ldots, g'_{s+m-n}) \circ (h_{m+1}, \ldots, h_n)$.

(2)′ $(g'_0, \ldots, g'_{s+m-n}) \in \mathscr{H}^{**}$.

(3)′ $g'_i g'_j \notin \mathscr{H} \cup \{1\}$ for any i and j with $i \neq j > s + m - n$.

Notice that if $m = 0$ then $s = n$, and then (1)′ and (2)′ are obviously true, while $h_i h_j \notin \mathscr{H} \cup \{1\}$ for any $i \neq j$, by [5, Lemma (3.6)(e)]. We may now assume that versions of (1)′ through (3)′ hold for $\mathbf{s}_{m-1}$. Thus, we have:

(1) $\mathbf{s}_{m-1} = (g_0, \ldots, g_{r+m-n-1}) \circ (h_m, \ldots, h_n)$.

(2) $(g_0, \ldots, g_{r+m-n-1}) \in \mathscr{H}^{**}$.

(3) $g_i g_j \notin \mathscr{H} \cup \{1\}$ for any i and j with $i \neq j > r + m - n - 1$.

Notice that $g_{m'} = h_m$. Since $\mathbf{s}_m$ is a simple distillation of $\mathbf{s}_{m-1}$ of codegree $n - m + 1$, there exist integers k and l with $0 \leq k \leq l \leq m'$, such that

$$\mathbf{s}_m = (g_0, \ldots, g_{k-1})^{h_m} \circ (h_m) \circ (g_l, \ldots, g_{m'-1}) \circ (h_{m+1}, \ldots, h_n).$$

Since $s = r - l + k$, we have $s + m - n = m' - l + k$ and so (1)′ holds. We have

$$(g'_0, \ldots, g'_{s+m-n}) = (g_0, \ldots, g_{k-1})^{h_m} \circ (h_m) \circ (g_l, \ldots, g_{m'-1}).$$

For any $i < m' - 1$, (2) says that $g_i g_{i+1} \notin \mathscr{H} \cup \{1\}$. Then also $(g_i g_{i+1})^{h_m} \notin \mathscr{H} \cup \{1\}$. Further, we have $(g_{k-1})^{h_m} h_m = (g_{k-1} h_m)^{h_m} \notin \mathscr{H} \cup \{1\}$ since (3) tells us that $g_{k-1} h_m \notin \mathscr{H} \cup \{1\}$. Since $h_m g_l = (g_l h_m)^{h_m^{-1}}$, (3) also implies that $h_m g_l \notin \mathscr{H} \cup \{1\}$. This proves (2)′.

Observe that for any $i \leq s + m - n$, there exists a chain of integers $c < c_1 < \cdots < c_u \leq m$ such that

$$g'_i = (h_c)^{h_{c_1} h_{c_2} \cdots h_{c_u}}.$$

Now consider the chain of $\mathscr{H}$-sequences

$$\mathbf{h} = \mathbf{h}_0 = \big(h_0, \dots, h_c, \dots, h_{c_1}, \dots, h_{c_u}, \dots, h_n\big)$$

$$\mathbf{h}_1 = \big(h_{c_1}\big) \circ \big(h_0, \dots, h_{c_1-1}\big)^{h_{c_1}} \circ \big(h_{c_1+1}, \dots, h_n\big)$$

$$\mathbf{h}_2 = \big(h_{c_1}, h_{c_2}\big) \circ \big(h_0, \dots, h_{c_1-1}\big)^{h_{c_1}h_{c_2}} \circ \big(h_{c_1+1}, \dots, h_{c_2-1}\big)^{h_{c_2}} \circ \big(h_{c_2+1}, \dots, h_n\big)$$

$$\vdots$$

$$\begin{aligned}\tilde{\mathbf{h}} &= \mathbf{h}_u \\ &= \big(h_{c_1}, h_{c_2}, \dots, h_{c_u}\big) \circ \big(h_0, \dots, h_{c_1-1}\big)^{h_{c_1} \cdots h_{c_u}} \circ \big(h_{c_1+1}, \dots, h_{c_2-1}\big)^{h_{c_2} \cdots h_{c_u}} \\ &\quad \circ \cdots \circ \big(h_{c_{u-1}+1}, \dots, h_{c_u-1}\big)^{h_{c_u}} \circ \big(h_{c_u+1}, \dots, h_n\big).\end{aligned}$$

This chain displays $\tilde{\mathbf{h}}$ as a braiding of $\mathbf{h}$. But $(h_c)^{h_{c_1} \cdots h_{c_u}}$ is an entry of $\tilde{\mathbf{h}}$, and also h_j is an entry of $\tilde{\mathbf{h}}$ for $j \geq m$. Then [5, Lemma (3.6)(e)] yields $(h_c)^{h_{c_1} \cdots h_{c_u}} h_j \notin \mathscr{H} \cup \{1\}$, and yields also $h_i h_j \notin \mathscr{H} \cup \{1\}$ if $m \leq i < j$. This completes the proof of $(3)'$.

Applying $(2)'$ to the case $m = n$, we obtain $\mathbf{h}' \in \mathscr{H}^{**}$. Now replace $\mathbf{h}$ with $\mathbf{h} \circ (h)$, and obtain a distillation of $\mathbf{h} \circ (h)$ via the sequence $\mathbf{s}_0 \circ (h), \dots, \mathbf{s}_n \circ (h)$, $(h) \circ \mathbf{s}_n = (h) \circ \mathbf{h}'$. Apply $(3)'$ to $\mathbf{s}_n \circ (h)$ and obtain $h'h \notin \mathscr{H} \cup \{1\}$ for any entry h' of $\mathbf{h}'$. As $hh' = (h'h)^{h^{-1}}$, the proof of the lemma is complete.

2.4. LEMMA. *Let* $\mathbf{h}$ *be an* $\mathscr{H}$*-sequence. Then there exists a braiding* $\mathbf{g}$ *of* $\mathbf{h}^{-1}$ *such that, for any distillation* $\mathbf{h}'$ *of* $\mathbf{h}$, $\mathbf{h}'^{-1}$ *is a distillation of* $\mathbf{g}$.

Proof. Notice first of all that if $\mathbf{s}$ is an $\mathscr{H}$-sequence and $\mathbf{t}$ is a simple braiding of $\mathbf{s}$, then $\mathbf{t}^{-1}$ is a simple braiding of $\mathbf{s}^{-1}$. It follows that the inverse of a braiding of $\mathbf{s}$ is a braiding of $\mathbf{s}^{-1}$.

Let $\mathbf{h} = (h_0, \dots, h_n)$. For any i, write $\overline{h}_i$ for h_i^{-1}, and put $g_i = \overline{h}_i^{h_{i+1} \cdots h_n}$. (This means that $g_n = h_n^{-1}$). Put $\mathbf{g} = (g_0, \dots, g_n)$. Then $\mathbf{g}^{-1}$ is a braiding of $\mathbf{h}$, as is displayed by the following sequence of intermediate braidings:

$$(h_0, \dots, h_n),$$

$$\big(h_1, h_0^{h_1}, h_2, \dots, h_n\big),$$

$$\big(h_2, h_1^{h_2}, h_0^{h_1h_2}, h_3, \dots, h_n\big),$$

$$\vdots$$

$$\big(h_n, h_{n-1}^{h_n}, \dots, h_0^{h_1 \cdots h_n}\big) = \mathbf{g}^{-1}.$$

The observation made in the first paragraph of this proof then says that $\mathbf{g}$ is a braiding of $\mathbf{h}^{-1}$.

For any $\mathscr{H}$-sequence $\mathbf{s}$ and for any j with $0 \le j \le l(s)$, the prefix $\mathbf{x}$ of $\mathbf{s}$ such that $l(\mathbf{x}) = l(\mathbf{s}) - t$ will be called the prefix of $\mathbf{s}$ of *colength* t.

We are given a distillation $\mathbf{h}'$ of $\mathbf{h}$. Thus, there exists a chain of $\mathscr{H}$-sequences

$$\mathbf{h} = \mathbf{s}_0, \ldots, \mathbf{s}_n = \mathbf{h}'$$

such that $\mathbf{s}_m$ is a simple distillation of $\mathbf{s}_{m-1}$ of codegree $n - m + 1$ $(1 \le m \le n)$. We may write

$$\mathbf{s}_{m-1} = (a_0, \ldots, a_r) \circ (h_m, \ldots, h_n)$$

for some $r < m$. Thus, $(a_0, \ldots, a_r)$ is the prefix of $\mathbf{s}_{m-1}$ of colength $n - m + 1$. Our object now will be to produce a chain

$$\mathbf{g} = \mathbf{t}_0, \ldots, \mathbf{t}_n = \mathbf{g}',$$

where $\mathbf{t}_m$ is a simple distillation of $\mathbf{t}_{m-1}$ of codegree $n - m + 1$ $(1 \le m \le n)$, and where $\mathbf{g}' = \mathbf{h}'^{-1}$. We claim that there exists such a chain, with the property that for every m with $0 \le m \le n$, the prefix of $\mathbf{t}_m$ of colength $n - m$ is $(\mathbf{x}^{-1})^{h_{m+1} \cdots h_n}$, where $\mathbf{x}$ is the prefix of $\mathbf{s}_m$ of colength $n - m$. For $m = n$, this simply says that $\mathbf{g}' = \mathbf{h}'^{-1}$, as desired. For $m = 0$, it says that $g_0 = h_0^{h_1 \cdots h_n}$, which is true by definition. Proceeding by induction, we may then assume that we have

$$\mathbf{t}_{m-1} = (b_0, \ldots, b_r) \circ (g_m, \ldots, g_n),$$
$$(b_0, \ldots, b_r) = (\bar{a}_r, \ldots, \bar{a}_0)^{h_m \cdots h_n}.$$

Since $\mathbf{s}_m$ is a simple distillation of $\mathbf{s}_{m-1}$ of codegree $n - m + 1$, we have

$$\mathbf{s}_m = (a_0, \ldots, a_{k-1})^{h_m} \circ (h_m) \circ (a_l, \ldots, a_r) \circ (h_{m+1}, \ldots, h_n)$$

for some k and l, with $0 \le k \le l \le r + 1$. Put $k' = r - l + 1$ and $l' = r - k + 1$. Then $0 \le k' \le l' \le r + 1$, and we then define

$$\mathbf{t}_m = (b_0, \ldots, b_{k'-1})^{g_m} \circ (g_m) \circ (b_{l'}, \ldots, b_r) \circ (g_{m+1}, \ldots, g_n).$$

We need to show

$$(b_0, \ldots, b_{k'-1})^{g_m} \circ (g_m) \circ (b_{l'}, \ldots, b_r)$$
$$= \Big((\bar{a}_r, \ldots, \bar{a}_l) \circ (\bar{h}_m) \circ (\bar{a}_{k-1}, \ldots, \bar{a}_0)^{h_m}\Big)^{h_{m+1} \cdots h_n}.$$

It will be left to the reader to perform this straightforward exercise, and to thereby complete the proof of (2.4).

We end this section by reviewing some of the terminology from [5], for the convenience of the reader.

Recall the definitions of b and $\mathscr{D}$ from (1.11).

2.5. DEFINITION. Let $\mathbf{h} = (h_0, \dots, h_n) \in \mathscr{H}^*$ and let $x \in X$. Put $x_0 = x$, and define $x_{k+1} = x_k \cdot h_k$, $0 \leq k \leq n$. Write z for x_{n+1}. Also, put $I_k = A(h_k) \cap [x_k, x_{k+1}] \cap [x_k, z]$. We say that *$x$ converges to z via h* (*relative to* $(b, \mathscr{D})$) if for all k, $0 \leq k \leq n$, we have

$$d(h_{k+1}, z) + a(h_k) \leq d(x_k, z) + b(h_k),$$
$$\text{with strict inequality if } \partial I_k \not\subseteq D. \quad (C_k)$$

We say that $\mathscr{H}$ has the *$(b, \mathscr{D})$-convergence property* if, whenever $\mathbf{h} \in \mathscr{H}^*$ and $x \in X$, there exists a braiding $\mathbf{h}'$ of $\mathbf{h}$ such that x converges to z via $\mathbf{h}'$. (Note that z is well-defined, by (2.0)(b).)

The main result of this paper, from which all else will follow, is that $\mathscr{H}$ has the $(b, \mathscr{D})$-convergence property. The proof of this will be achieved in Section 5, below.

2.6. DEFINITION. Let $\mathscr{P}$ be a mapping which associates to each $\mathbf{h} \in \mathscr{H}^*$ a collection $\mathscr{P}(\mathbf{h})$ of closed oriented segments of X. We say that $\mathscr{P}$ is a *polarization* (*of X over $\mathscr{H}$*) if the following conditions hold.

(a) $\mathscr{P}((\emptyset))$ is the set of all degenerate closed segments of X. That is, $\mathscr{P}((\emptyset)) = X$.

(b) $\mathscr{P}(\mathbf{h}) = \mathscr{P}(\mathbf{h}')$ if $\mathbf{h}'$ is a braiding of $\mathbf{h}$.

(c) If $J \in \mathscr{P}(\mathbf{h})$ then $J^{opp} \in \mathscr{P}(\mathbf{h}^{-1})$, and $J \cdot g \in \mathscr{P}(\mathbf{h}^g)$ for any $g \in \langle \mathscr{H} \rangle$.

2.7. DEFINITION. Let $\mathscr{P}$ be a fixed polarization of X over $\mathscr{H}$, let $\mathbf{h} \in \mathscr{H}^*$, and let $J \in \mathscr{P}(\mathbf{h})$. Then $\mathscr{E}(\mathbf{h}, J)$ denotes the set of all $h \in \mathscr{H}$ satisfying the following three conditions.

(a) $\mathbf{h} \circ (h) \in \mathscr{H}^*$,

(b) $J \cap A(h) \neq \emptyset$, and

(c) if J is non-degenerate, then $J \cap A(h)$ is non-degenerate, and is oriented toward $\partial_0 J$ by h.

2.8. DEFINITION. Let $\mathscr{P}$ be a polarization of X over $\mathscr{H}$, and let b and $\mathscr{D}$ be defined as in (1.11). We say that *$\mathscr{P}$ controls convergence* (*relative to* $(b, \mathscr{D})$) if the following conditions hold whenever we have $\mathbf{h} \in \mathscr{H}^*$, $J \in \mathscr{P}(\mathbf{h})$, and $h \in \mathscr{E}(\mathbf{h}, J)$.

(1) (*Exclusion*)

(a) $J \not\subseteq A(h)$ if $\mathbf{h} \neq (\emptyset)$,

(b) $l(J \cap A(h)) < a(h)$, and

(c) if $\partial J \cap A(h) \neq \emptyset$ then $b(h) > 2 \cdot l(J \cap A(h))$.

(2) (*Extension*)

(a) We have $[(\partial_0 J) \cdot h, \partial_1 J] \in \mathscr{P}(\mathbf{h} \circ (h))$, provided that:

(i) $\partial J \cap A(h) = \emptyset$, and

(ii) $l(J \cap A(h)) \geq a(h) - \frac{1}{2}b(h)$, with strict inequality if $\partial(J \cap A(h)) \not\subseteq D$.

(b) For $x \in A(h)$, we have $[x \cdot h, \partial_1 J] \in \mathscr{P}(\mathbf{h} \circ (h))$, provided that:

(i) $\partial_0 J \in [x, x \cdot h)$, and

(ii) $d(\partial_0 J, x \cdot h) \geq \frac{1}{2}b(h)$, with strict inequality if $\{\partial_0 J, x\} \subseteq D$.

3. A POLARIZATION OF X

Let $J = [x, x']$ be a non-degenerate, oriented closed segment of X. Recall from [5, Sect. 2] that this means that J consists of a metric morphism $J\colon [0, d(x, x')] \to X$ with $0 \cdot J = x$ and with $d(x, x') \cdot J = x'$. We also write $\partial_0 J$ for x and $\partial_1 J$ for x', and we often abuse notation and identify the morphism J with its image in X. By a *partition* of J we will mean a sequence (possibly empty) of integers $a_1, \ldots, a_n$, with $0 < a_1 \leq \cdots \leq a_n < d(x, x')$. Setting $y_i = a_i \cdot J$, we will then write

$$x \leq y_1 \leq \cdots \leq y_n \leq x'. \tag{3.1}$$

(Although we allow $y_i = y_{i+1}$, later on we will impose conditions which will force $y_{i-1} \neq y_{i+1}$ for $1 < i < n$, and which will force $x \neq y_1$ and $y_n \neq x'$.)

Let a partition of J be given as in (3.1), and let $(x_0, \ldots, x_{n+1})$ be a sequence of points of X, satisfying the following conditions.

(3.2)

(a) $x_0 = x$ and $x_{n+1} = x'$.

(b) For all i, $1 \leq i \leq n$, we have:

(i) $x_i \notin J$ and $[x_i, y_i]$ is the bridge from x_i to J, and

(ii) $y_i = Y(x_{i-1}, x_i, x_{i+1})$.

Define $\Sigma = \Sigma(x_0, \ldots, x_{n+1})$ to be the smallest subtree of X containing $\{x_0, \ldots, x_{n+1}\}$. Thus:

$$\Sigma = \bigcup_{i,j} [x_i, x_j] = J \cup \left(\bigcup_{i=1}^{n} [x_i, y_i] \right).$$

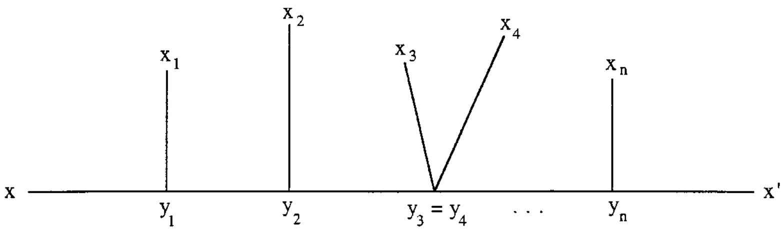


FIGURE 3

We say that Σ is a *simplex over* J (*relative to the partition* $(x \leq y_1 \leq \cdots \leq y_n \leq x')$); see Fig. 3.

Recall that $\mathscr{H}^{**}$ denotes the set of all reduced $\mathscr{H}$-sequences.

3.3. DEFINITION. Let $\mathbf{h} = (h_0, \ldots, h_n) \in \mathscr{H}^{**}$, and let $\Sigma = \Sigma(x_0, \ldots, x_{n+1})$ be a simplex over J. We say that $\mathbf{h}$ and Σ are *engaged* if, for all i with $0 \leq i \leq n$, we have $\Sigma \cap A(h_i) = [x_i, x_{i+1}]$, oriented toward x_i by h_i.

If $\mathbf{h}$ and Σ are engaged, as shown in Fig. 4, we put

$$J_i = J \cap A(h_i),$$
$$d_i = l(J_i),$$
$$m_i = m(h_i) \qquad (\text{see } (1.9)).$$

For any segment $\mathbf{s} = (h_k, \ldots, h_l)$ of $\mathbf{h}$, we put also

$$\begin{aligned} d(\mathbf{s}) &= (d_k, \ldots, d_l), \\ m(\mathbf{s}) &= (m_k, \ldots, m_l). \end{aligned} \tag{3.4}$$

3.5. LEMMA. *Let* $\mathbf{h} = (h_0, \ldots, h_n) \in \mathscr{H}^{**}$, *and let* $\Sigma = \Sigma(x_0, \ldots, x_{n+1})$ *be a simplex over* J. *Suppose that* $\mathbf{h}$ *and* Σ *are engaged. Then the following hold for all* i, $0 < i \leq n + 1$.

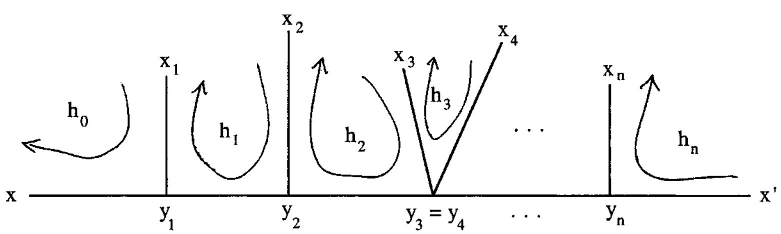


FIGURE 4

(a) $\Sigma \cap A(h_{i-1}) \cap A(h_i) = [x_i, y_i]$.

(b) $d(x_i, y_i) \leq 2$, *with strict inequality unless* $\{x_i, y_i\} \subseteq D$.

Proof. Part (a) is by construction. Since $h_{i-1}h_i \notin \mathscr{H} \cup \{1\}$, part (b) then follows from (1.8).

We will be concerned with segments $\mathbf{s}$ of $\mathbf{h}$ which, to begin with, satisfy the following condition.

$(*)$ We have $\mathbf{s} = (h_k, \ldots, h_l)$, $0 < k$, $l < n$, and $l(\Sigma \cap A(h_{i-1}) \cap A(h_i)) = d(x_i, y_i) = 2$ for all $i, k \leq i \leq l+1$.

3.6. Definition. Let $\mathbf{h} \in \mathscr{H}^{**}$, let $\mathbf{s} = (h_k, \ldots, h_l)$ be a segment of $\mathbf{h}$, and let Σ be a simplex over J. Assume that $\mathbf{h}$ and Σ are engaged.

(I) We say that $\mathbf{s}$ is *special*, *of type* I, provided that $(*)$ holds, and provided that

$$l(\mathbf{s}) = 1 \text{ or } 3,$$

$$m(\mathbf{s}) = (2) \text{ or } (2,2,2),$$

and

$$d(\mathbf{s}) = (4) \text{ or } (2,4,2).$$

(II) We say that $\mathbf{s}$ is *special, of type* II, provided that $(*)$ holds, and $m(\mathbf{s})$ is an alternating string of 2's and 3's, and one of the following holds.

(i) $l(\mathbf{s}) = 1$, $m(\mathbf{s}) = (3)$, and $d(\mathbf{s}) = (6)$.

(ii) $l(\mathbf{s}) = 2$, $m(\mathbf{s}) = (2,3)$ or $(3,2)$, and $d(\mathbf{s}) = (4,4)$.

(iii) $l(\mathbf{s}) = 3$, $m(\mathbf{s}) = (2,3,2)$, and $d(\mathbf{s}) = (4,2,4)$.

(iv) $l(\mathbf{s}) > 4$, and $d(\mathbf{s}) = \lambda \circ \xi_t \circ \mu$, where $t = [(l-k+1)/4]$,

$$\lambda = \begin{cases} (4,2) & \text{if } m_k = 3 \\ (4,2,2) & \text{if } m_k = 2 \end{cases} \qquad \mu = \begin{cases} (2,4) & \text{if } m_l = 3 \\ (2,2,4) & \text{if } m_l = 2, \end{cases}$$

and where ξ_t is defined inductively by

$$\xi_1 = (6), \qquad \text{and} \qquad \xi_i = \xi_{i-1} \circ (2,2,2,6).$$

3.7. *Remark.* It follows from (3.6) that $l(\mathbf{s})$ is not divisible by 4. Moreover, if $\mathbf{s}$ is of type II and $l(\mathbf{s}) > 4$, then either

$$l(\mathbf{s}) \equiv 1 \pmod 4, \qquad \text{and} \qquad m(\mathbf{s}) = (3,2,\ldots,2,3),$$

$$l(\mathbf{s}) \equiv 2 \pmod 4,$$

or

$$l(\mathbf{s}) \equiv 3 \pmod 4, \qquad \text{and} \qquad m(\mathbf{s}) = (2,3,\ldots,3,2).$$

Next, let $\mathbf{x}$ be a segment of $\mathbf{h}$ of the form

$$\mathbf{x} = \left(h_{k_1}\right) \circ \mathbf{s}_1 \circ \left(h_{k_2}\right) \circ \cdots \circ \mathbf{s}_u \circ \left(h_{k_{u+1}}\right),$$

where each $\mathbf{s}_i$ is special (of type I or II) relative to Σ. Write

$$\mathbf{s}_i = \left(h_{r_i}, \ldots, h_{s_i}\right), \qquad 1 \le i \le u,$$

so that $r_i = k_i + 1$ and $s_i = k_{i+1} - 1$. Define integers $p(\mathbf{s}_i)$ and $q(\mathbf{s}_i)$, as

$$\begin{aligned} &\left(p(\mathbf{s}_i), q(\mathbf{s}_i)\right) = (2, 2) \qquad \text{if } \mathbf{s}_i \text{ is of type I} \\ &\left(p(\mathbf{s}_i), q(\mathbf{s}_i)\right) = \left(8 - 2 \cdot m_{k_i+1}, 8 - 2 \cdot m_{k_{i+1}-1}\right) \qquad \text{if } \mathbf{s}_i \text{ is of type II.} \end{aligned} \tag{3.8}$$

3.9. Definition. Let $\mathbf{h}$ and Σ be engaged. A segment $\mathbf{x} = (h_{k_1}) \circ \mathbf{s}_1 \circ (h_{k_2}) \circ \cdots \circ \mathbf{s}_u \circ (h_{k_{u+1}})$ of $\mathbf{h}$ will be said to be *extraspecial* (relative to Σ) if $u \ge 1$ and:

(a) Each $\mathbf{s}_i$ is a special segment of type I or II;

(b) For all i with $1 \le i \le u$, we have $m_{k_i} \ge 6$ and $m_{k_{i+1}} \ge 6$, unless $\mathbf{s}_i$ is of type I and of length 1, in which case we have

$$1/m_{k_i} + 1/m_{k_{i+1}} \le 1/2;$$

(c) We have the inequalities

$$\begin{aligned} &l\left(\Sigma \cap A\left(h_{k_1}\right)\right) \ge 2 \cdot m_{k_1} - p(\mathbf{s}_1), \\ &l\left(\Sigma \cap A\left(h_{k_i}\right)\right) \ge 2 \cdot m_{k_i} - p(\mathbf{s}_1) - q(\mathbf{s}_{i-1}) \qquad (1 < i < u + 1), \\ &l\left(\Sigma \cap A\left(h_{k_{u+1}}\right)\right) \ge 2 \cdot m_{k_{u+1}} - q(\mathbf{s}_u). \end{aligned}$$

Moreover, for each i, the corresponding inequality is strict if $\partial(\Sigma \cap A(h_{k_i})) \subseteq D$.

3.10. Definition. A segment $\mathbf{a}$ of $\mathbf{h}$ will be said to be *ordinary* (relative to Σ) if for every segment (h) of $\mathbf{a}$ of length 1 we have

$$l(\Sigma \cap A(h)) \ge 2 \cdot m(h)$$

with strict inequality if $\partial(\Sigma \cap A(h)) \subseteq D$. (In particular, the empty segment $(\varnothing)$ is ordinary.)

3.11. Definition. Let $\mathbf{h} \in \mathscr{H}^{**}$, and let Σ be a simplex over J. We say that $\mathbf{h}$ and Σ are married if $\mathbf{h}$ and Σ are engaged, and if we can write

$$\mathbf{h} = \mathbf{a}_1 \circ \mathbf{x}_1 \circ \cdots \circ \mathbf{a}_m \circ \mathbf{x}_m \circ \mathbf{a}_{m+1},$$

where each $\mathbf{a}_i$ is an ordinary segment, and each $\mathbf{x}_j$ is an estraspecial segment of $\mathbf{h}$, relative to Σ.

We may now, at long last, define a suitable polarization of X over $\mathscr{H}$. The reader will need to recall the notion of "distillation" of an $\mathscr{H}$-sequence, from (2.2).

3.12. Definition. Let J be an oriented closed segment of X, and let $\mathbf{h}' \in \mathscr{H}^*$. We declare

$$J \in \mathscr{P}(\mathbf{h}')$$

if there exists a simplex Σ over J, a braiding $\mathbf{h}''$ of $\mathbf{h}'$, and a distillation $\mathbf{h}$ of $\mathbf{h}''$, such that $\mathbf{h}$ and Σ are married.

3.13. Proposition. *$\mathscr{P}$ is a polarization of X over $\mathscr{H}$.*

Proof. First of all, it is clear from the definitions (3.2) and (3.3) that the only simplices which are engaged with the empty sequence, $(\varnothing) \in \mathscr{H}^*$, are the simplices which consist of a single vertex. Thus, $\mathscr{P}((\varnothing))$ is the set of all degenerate closed segments of X, as required in Definition (2.6). Next, it is immediate from (3.12) that $\mathscr{P}(\mathbf{h}') = \mathscr{P}(\mathbf{h}'')$ for all braidings $\mathbf{h}''$ of $\mathbf{h}'$. Fix such a braiding $\mathbf{h}''$, and let $\mathbf{h}$ be a distillation of $\mathbf{h}''$. Then $\mathbf{h} \in \mathscr{H}^{**}$, by (2.3). For any $g \in \langle \mathscr{H} \rangle$ one sees that $\mathbf{h}''^{g}$ is a braiding of $\mathbf{h}'^{g}$, and $\mathbf{h}^g$ is a distillation of $\mathbf{h}''^{g}$. Let $J \in \mathscr{P}(\mathbf{h}')$, and let Σ be a simplex over J such that $\mathbf{h}$ and Σ are married. Then $\Sigma \cdot g$ is a simplex over $J \cdot g$, and it is plain that $\mathbf{h}^g$ and $\Sigma \cdot g$ are married. Thus, $J \cdot g \in \mathscr{P}(\mathbf{h}'^{g})$. Finally, (2.4) shows that there is a braiding $\mathbf{g}$ of $\mathbf{h}'^{-1}$ such that $\mathbf{h}^{-1}$ is a distillation of $\mathbf{g}$. Evidently, $\mathbf{h}^{-1}$ and J^{opp} are married, so we have $J^{opp} \in \mathscr{P}(\mathbf{h}'^{-1})$. Thus, all the requirements of Definition (2.6) have been met by $\mathscr{P}$.

4. PROPERTIES OF $\mathscr{P}$

Let $\mathscr{P}$ be the polarization of X over $\mathscr{H}$ which was introduced in the preceding section. Our eventual goal is to show that $\mathscr{P}$ controls convergence relative to $(b, \mathscr{D})$, in the sense of Definition (2.8). This will be achieved in Section 5. The aim of the present section is to supply a few supporting results for the arguments in Section 5.

Fix $\mathbf{h}' \in \mathscr{H}^*$, and fix $J \in \mathscr{P}(\mathbf{h}')$. Thus, there is a braiding $\mathbf{h}''$ of $\mathbf{h}'$, a distillation $\mathbf{h}$ of $\mathbf{h}''$, and a simplex Σ over J, such that $\mathbf{h}$ and Σ are married, in the sense of (3.11). Write

$$\begin{aligned}
\mathbf{h} &= (h_0, \dots, h_n), \\
\Sigma &= \Sigma(x_0, \dots, x_{n+1}), \\
J &= [x, x'], \qquad \text{partitioned by } x \le y_1 \le \cdots \le y_n \le x', \text{where } x = x_0, \\
& \qquad\qquad x' = x_{n+1}, \qquad \text{and } y_i = Y(x_{i-1}, x_i, x_{i+1}), \\
J_i &= J \cap A(h_i), \\
d_i &= l(J_i), \\
m_i &= m(h_i).
\end{aligned}$$

Fix a decomposition of $\mathbf{h}$ as in (3.11),

$$\mathbf{h} = \mathbf{a}_1 \circ \mathbf{x}_1 \circ \mathbf{a}_2 \circ \cdots \circ \mathbf{a}_m \circ \mathbf{x}_m \circ \mathbf{a}_{m+1}, \tag{4.1}$$

where each $\mathbf{a}_i$ is an ordinary segment (possibly empty), and each $\mathbf{x}_j$ is an extraspecial segment of $\mathbf{h}$, relative to Σ. For $\mathbf{x} \in \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ we can write, as in (3.9),

$$\mathbf{x} = (h_{k_1}) \circ \mathbf{s}_1 \circ (h_{k_2}) \circ \cdots \circ \mathbf{s}_u \circ (h_{k_{u+1}}), \tag{4.2}$$

where each $\mathbf{s}_i$ is a special segment (of type I or II) relative to Σ, and where the short segments (h_{k_i}) satisfy the various conditions in (3.9). The segments $\mathbf{x}_j$, $1 \le j \le m$, will be called the *basic extraspecial segments* of $\mathbf{h}$. A segment $\mathbf{s}$ of $\mathbf{h}$ will be called a *basic special segment* if $\mathbf{s} = \mathbf{s}_i$ in some basic extraspecial segment $\mathbf{x}$, as in (4.2). We make no attempt to decide whether the decompositions (4.1) and (4.2) are uniquely determined.

We now define a partition $\{\mathscr{S}_j\}_{0 \le j \le 3}$ of $\{0, \dots, n+1\}$, as

$$\begin{aligned}
\mathscr{S}_0 &= \{k : (h_k) \text{ is a segment of some } \mathbf{a}_i\} \\
\mathscr{S}_1 &= \{k : (h_k) = (h_{k_1}) \text{ or } (h_k) = (h_{k_{u+1}}) \\
&\qquad\qquad \text{in some basic extraspecial } \mathbf{x} \text{ as in } (4.2)\} \\
\mathscr{S}_2 &= \{k : (h_k) = (h_{k_i}), \text{with } 1 < i < u+1, \\
&\qquad\qquad \text{in some basic extraspecial } \mathbf{x} \text{ as in } (4.2)\} \\
\mathscr{S}_3 &= \{k : (h_k) \text{ is a segment of some basic special segment of } \mathbf{h}\}.
\end{aligned}$$

We mention that by (h_k) we mean the $(k+1)$st segment of length 1 in $\mathbf{h}$. It is entirely possible that $h_j = h_k$ for two distinct indices j and k.

4.3. LEMMA. *Suppose* $\mathbf{h}' \neq (\varnothing)$. *Then* $\mathbf{h} \neq (\varnothing)$ *and* J *is non-degenerate. Indeed, for both* $k = 0$ *and* n, J_k *contains at least two points not in* D.

Proof. It is an obvious consequence of Definition (2.2) that $\mathbf{h} \neq (\varnothing)$. Suppose now that J_0 contains at most one point which is not in D. Then $l(J_0) \leq 2$, with equality only if $\partial J_0 = \{x, y_1\} \subseteq D$. Then also (3.5) yields $l(\Sigma \cap A(h_0)) = d(x, x_1) \leq 4$, with equality only if $\{x, x_1\} \subseteq D$. On the other hand, $\mathbf{h}$ and Σ are married, so $0 \in \mathscr{S}_0 \cup \mathscr{S}_1$. If $0 \in \mathscr{S}_0$ then, by definition, we have $d(x, x_1) \geq 2 \cdot m_0$, with strict inequality if $\{x, x_1\} \subseteq D$. Thus, $0 \in \mathscr{S}_1$, and (3.9)(c) gives $d(x, x_1) \geq 2 \cdot m_0 - p(\mathbf{s}_1)$, where $\mathbf{s}_1$ is the basic special sequence to the right of (h_0) in $\mathbf{h}$. Again, the inequality is strict if $\{x, x_1\} \subseteq D$. Here (3.9)(b) yields $2 \cdot m_0 - p(\mathbf{s}_1) \geq 6$ unless $m_0 = 3$ and $p(\mathbf{s}_1) = 2$. Thus, $m_0 = 3$, $p(\mathbf{s}_1) = 2$, and $d(x, x_1) = 4$. But then $\{x, x_1\} \subseteq D$ and $d(x, x_1) > 4$, a contradiction.

4.4. LEMMA. *Let* $0 \leq k \leq n$ *and suppose* $d_k \leq 2$. *Then* $0 < k < n$ *and one of the following holds.*

(i) $k \in \mathscr{S}_0$, $d_k = 2$, *and* $m_k = 2$.

(ii) $k \in \mathscr{S}_1$, $d_k = 2$, $m_k = 3$, *and the basic special segment to one side of* (h_k) *is of type I and of length* 1.

(iii) $k \in \mathscr{S}_2$, $d_k = 2$, $m_k = 6$, $m_{k-1} = m_{k+1} = 2$, $d_{k-1} = d_{k+1} = 4$, *and the special segments on each side of* (h_k) *in* $\mathbf{h}$ *are of type II*.

(iv) $k \in \mathscr{S}_2$, $d_k = 0$ *or* 2, *and the special segments on each side of* (h_k) *in* $\mathbf{h}$ *are of type I and of length* 1. *If* $d_k = 2$ *then* $m_k = 3$ *or* 4, *and if* $d_k = 0$ *then* $m_k = 3$.

(v) $k \in \mathscr{S}_3$, $d_k = 2$, *and* $m_k = 2$ *or* 3. *Further, if* $m_k = 3$ *then* $\{k-1, k+1\} \subseteq \mathscr{S}_3$ *and* $m_{k-1} = m_{k+1} = 2$.

Proof. Suppose first that $k \in \mathscr{S}_0$. Then (3.10) says that $l(\Sigma \cap A(h_k)) = d(x_k, x_{k+1}) \geq 2 \cdot m_k$, with strict inequality if $\{x_k, x_{k+1}\} \subseteq D$. Since $d_k \leq 2$, (3.5) yields $d(x_k, x_{k+1}) \leq 6$, with equality only if $\{x_k, y_k, y_{k+1}, x_{k+1}\} \subseteq D$. Thus, $m_k = 2$ and $d(x_k, x_{k+1}) \geq 4$. If $k = 0$ or n, (3.5) yields $d(x_k, x_{k+1}) = 4$ and $\{x_k, x_{k+1}\} \subseteq D$; a contradiction. Thus, $0 < k < n$. Also, if $d_k < 2$ then (3.5) yields $d(x_k, x_{k+1}) \leq 4$, with equality only if $\{x_k, x_{k+1}\} \subseteq D$. Again, this is inconsistent with (3.10), so $d_k = 2$ and (i) holds.

Suppose next that $k \in \mathscr{S}_1$, so that (h_k) comes either at the beginning or at the end of a basic extraspecial segment $\mathbf{x}$ of $\mathbf{h}$. By symmetry, we may assume without loss that (h_k) is at the beginning of $\mathbf{x}$. In the notation of (4.2), $(h_k) = (h_{k_1})$, and $\mathbf{s}_1$ is the basic special sequence following (h_k). We apply (3.9)(c) to $\mathbf{x}$, and obtain $d(x_k, x_{k+1}) \geq 2 \cdot m_k - p(\mathbf{s}_1)$, with strict inequality if $\{x_k, x_{k+1}\} \subseteq D$. Here $p(\mathbf{s}_1)$ is given by (3.8). Thus, $p(\mathbf{s}_1) = 2$ or else $\mathbf{s}_1$ is of type II and $p(\mathbf{s}_1) = 4$. Since (3.5) gives $d(x_k, x_{k+1}) \leq 6$, we then have $m_k \leq 5$. Then (3.9)(b) says that $\mathbf{s}_1$ is of type I and length 1, and

$m_k > 2$. Here $p(\mathbf{s}_1) = 2$, so $m_k = 3$ or 4. Further, if $m_k = 4$ then $d(\mathbf{x}_k, x_{k+1}) = 6$, and (3.5) yields $\{x_k, x_{k+1}\} \subseteq D$, and then (3.9)(c) gives $d(x_k, x_{k+1}) > 6$, a contradiction. Thus, $m_k = 3$, $2 \cdot m_k - p(\mathbf{s}_1) = 4$, and so we have $d(x_k, x_{k+1}) \geq 4$, with strict inequality if $\{x_k, x_{k+1}\} \subseteq D$. This forces $k \neq 0$ (so that the first assertion in (4.4) is now proven), and it forces also $d_k = 2$, in view of (3.5). Thus, (ii) holds.

Suppose next that $k \in \mathscr{S}_2$. Again let $\mathbf{x}$ be the basic extraspecial segment involving (h_k). In the notation of (4.2) we now have $k = k_i$ for some i, $1 < i < u + 1$, and $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ are the basic special sequences which, respectively, precede and follow (h_k) in $\mathbf{x}$. Now (3.9)(c) says

$$d(x_k, x_{k+1}) \geq 2 \cdot m_k - p(\mathbf{s}_i) - q(\mathbf{s}_{i-1}) \tag{$*$}$$

with the familiar condition on strict inequality. Since also $d(x_k, x_{k+1}) \leq 6$, with strict inequality if $\{x_k, x_{k+1}\} \subseteq D$, we get

$$6 > 2 \cdot m_k - p(\mathbf{s}_i) - q(\mathbf{s}_{i-1}). \tag{$**$}$$

Since $p(\mathbf{s}_i) \leq 4$ and $q(\mathbf{s}_{i-1}) \leq 4$, by (3.8), we conclude that $m_k \leq 6$.

Suppose $m_k = 6$. Then $p(\mathbf{s}_i) = q(\mathbf{s}_{i-1}) = 4$, and (3.8) says that $m_{k-1} = m_{k+1} = 2$, and both $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ are of type II, and $d_{k-1} = d_{k+1} = 4$. Now $2 \cdot m_k - p(\mathbf{s}_i) - q(\mathbf{s}_{i-1}) = 4$. If $d_k < 2$ then (3.5) gives $d(x_k, x_{k+1}) \leq 4$, with equality only if $\{x_k, x_{k+1}\} \subseteq D$. Thus, $d_k = 2$ and (iii) holds.

Now suppose $m_k < 6$. Then (3.9)(b) says that $\mathbf{s}_{i-1}$ and $\mathbf{s}_i$ are special of type I and length 1, and (3.8) gives $p(\mathbf{s}_i) = q(\mathbf{s}_{i-1}) = 2$. Also, (3.9)(b) says that $m_k > 2$, and then $(**)$ yields $m_k = 3$ or 4. If $d_k = 2$ then (iv) holds, so assume $d_k = 0$ or 1. As we have seen, (3.5) then gives $d(x_k, x_{k+1}) \leq 4$, with equality only if $\{x_k, x_{k+1}\} \subseteq D$. Then $(*)$ yields $m_k = 3$. In order to obtain (iv) it now suffices to show that $d_k \neq 1$. But if $d_k = 1$ then either $\mathbf{s}_{i-1}$ or $\mathbf{s}_i$ fails to satisfy the condition $(*)$ preceding, and involved in, Definition (3.6). Thus, $d_k \neq 1$ and (iv) holds.

Finally, assume that $k \in \mathscr{S}_3$. Then (3.6) says that $d_k = 2$ and $m_k = 2$ or 3. Assume further that $m_k = 3$, and let $\mathbf{s}$ be the basic special segment in $\mathbf{h}$ involving (h_k). Then $\mathbf{s}$ is of type II, and inspection of (3.6) reveals that, with $d_k = 2$, h_k does not occur as either the first or last term in the sequence $\mathbf{s}$. Thus, we have $k \pm 1 \in \mathscr{S}_3$ and $m_{k\pm 1} = 2$, so that (v) holds.

4.5. LEMMA. *Assume that* $d_k = d_{k+1} = 2$, *and that* $m_k + m_{k+1} > 4$. *Then one of the following holds.*

(i) $k \in \mathscr{S}_1$, $k + 1 \in \mathscr{S}_0$, $m_k = 3$, $m_{k+1} = 2$, *and* (h_{k-1}) *is a basic special segment of type I.*

(ii) $k \in \mathscr{S}_0$, $k + 1 \in \mathscr{S}_1$, $m_k = 2$, $m_{k+1} = 3$, *and* (h_{k+2}) *is a basic special segment of type I.*

(iii) $\{k, k+1\} \subseteq \mathscr{S}_1$, $m_k = m_{k+1} = 3$, *and both* (h_{k-1}) *and* (h_{k+2}) *are basic special segments of type* I.

(iv) $\{k, k+1\} \subseteq \mathscr{S}_3$, *and* (h_k, h_{k+1}) *is part of a basic special segment of type* II.

Proof. Suppose $k \in \mathscr{S}_0$. If also $k+1 \in \mathscr{S}_0$ then $m_k + m_{k+1} = 4$ by (4.4)(i). Thus, we must have $k+1 \in \mathscr{S}_1$, and then (ii) follows from outcomes (i) and (ii) of (4.4). Similarly, if $k+1 \in \mathscr{S}_0$ then we get (i). So, we may now assume $k \notin \mathscr{S}_0$ and $k+1 \notin \mathscr{S}_0$. If $\{k, k+1\} \subseteq \mathscr{S}_1$ then (iii) follows from (4.4)(ii). If $k \in \mathscr{S}_2$ then $d_{k+1} = 4$ by (4.4)(iii), so we now conclude that $\{k, k+1\} \subseteq \mathscr{S}_3$. Then (h_k, h_{k+1}) is part of a basic special sequence $\mathbf{s}$ in $\mathbf{h}$. Since $m_k + m_{k+1} > 4$, $\mathbf{s}$ is then of type II, and (iv) holds.

4.6. Lemma. *Assume that* $d_{k-1} = d_k = d_{k+1} = 2$, *and assume that both* $m_{k-1} + m_k > 4$ *and* $m_k + m_{k+1} > 4$. *Then one of the following holds.*

(i) $k \in \mathscr{S}_0, \{k-1, k+1\} \subseteq \mathscr{S}_1, m_{k-1} = m_{k+1} = 3$, *and both* (h_{k-2}) *and* (h_{k+2}) *are basic special segments of type* I.

(ii) $\{k-2, k-1, k, k+1, k+2\} \subseteq \mathscr{S}_3$, $d_{k-2} = d_{k+2} = 6$, $m_{k-1} = m_{k+1} = 2$, *and* $m_{k-2} = m_k = m_{k+2} = 3$.

Proof. If $k \in \mathscr{S}_0$ we get (i) as a consequence of (4.5)(i), (ii). If $k \in \mathscr{S}_1$ then (4.5) shows that $\{k-1, k+1\} \subseteq \mathscr{S}_0 \cup \mathscr{S}_1$, which is contrary to the definition of $\mathscr{S}_1$. Now (4.5) forces $k \in \mathscr{S}_3$, and then also (h_{k-1}, h_k, h_{k+1}) is part of a basic special sequence $\mathbf{s}$ of type II. Since $d_{k-1} = d_k = d_{k+1} = 2$, a glance at (3.6)(II) shows that $d_{k-2} = d_{k+2} = 6$, $m_{k-2} = m_{k+2} = 3$, and $(h_{k-2}, \ldots, h_{k+2})$ is part of $\mathbf{s}$. Thus, (ii) holds.

For the remainder of this section, and in the following section, we fix an element h of $\mathscr{E}(\mathbf{h}', J)$. Recall from Definition (2.7) that this means that $\mathbf{h}' \circ (h) \in \mathscr{H}^*$, and $J \cap A(h)$ is a closed segment, oriented toward x' by h, and non-degenerate if J is non-degenerate. Put

$$I = J \cap A(h) = [u, v]$$

with $[u, v] \leq [x, x']$. (That is to say, $d(u, x) \leq d(v, x)$ and $d(u, x') \geq d(v, x')$.) Thus, I is oriented toward v by h. Put

$$\mathscr{S}_4 = \{k : y_k \in [u, v], u \neq y_k \neq v, 1 \leq k \leq n\}.$$

That is, $k \in \mathscr{S}_4$ if y_k is in the "interior" of I.

4.7. Lemma. *For any* k *with* $0 \leq k \leq n$, $A(h) \cap A(h_k)$ *contains at most one vertex not in* D. *In particular,* $l(I \cap J_k) \leq 2$.

Proof. This is immediate from (2.3) and (1.8).

4.8. LEMMA. *Let $k \in \mathscr{S}_4$, and suppose $y_k \neq y_{k+1}$. Then* $1/m_k + 1/m_{k+1} + 1/m(h) \leq 1$.

Proof. This is immediate from (2.3) and (1.10).

4.9. LEMMA. *Suppose $l(I) \geq \frac{1}{2}a(h)$. Then $\partial J \cap I = \emptyset$, $\{u, y_r, \dots, y_s, v\} \subseteq D$, and one of the following holds.*

(i) $\mathscr{S}_4 = \{r\}$, $a(h) = 8$, *and* $d(u, y_r) = d(y_r, v) = 2$.

(ii) $\mathscr{S}_4 = \{r, r+1\}$, $y_r = y_{r+1}$, $a(h) = 8$, *and* $d(u, y_r) = d(y_{r+1}, v) = 2$.

(iii) $\mathscr{S}_4 = \{r, r+1\}$, $y_r \neq y_{r+1}$, $a(h) = 12$, *and* $d(u, y_r) = d(y_r, y_{r+1}) = d(y_{r+1}, v) = 2$.

Moreover, we either have $r \in \mathscr{S}_0$ and $m_r = 2$, or else $r \in \mathscr{S}_2$ and $m_r = 6$.

Proof. Suppose $x \in I$. As $l(I) > 2$, it then follows from (4.3) that $I \cap J_0$ contains two points not in D, and this contradicts (4.7). Thus, $x \notin I$, and similarly $x' \notin I$. The same reasoning shows also that $I \not\subseteq J_k$ for any k, $0 \leq k \leq n$, and hence $\mathscr{S}_4 \neq \emptyset$.

Suppose next that $y_k = y_{k+1}$ for some k, with $r \leq k < s$. Then $d_k = 0$, so that (h_{k-1}) and (h_{k+1}) are basic special sequences, of type I, by (4.4). Consulting Definition (3.6), this means that $d_{k-1} = d_{k+1} = 4$. Then (4.7) forces $u \in J_{k-1}$, $v \in J_{k+1}$, with each of $[u, y_k]$ and $[y_{k+1}, v]$ containing at most one point not in D. Since $a(h) \geq 8$, by (1.12), it then follows that $a(h) = 8$, $m(h) = 2$, $d(u, y_k) = d(y_{k+1}, v) = 2$, and $\{u, y_k, v\} \subseteq D$. Thus, (ii) holds in this case.

Suppose next that $r = s$. Thus, $u \in J_{r-1}$ and $v \in J_r$. As in the preceding paragraph, each of $[u, y_k]$ and $[y_k, v]$ contains at most one point not in D, and since $a(h) \geq 8$, (i) holds in this case.

We may now assume $r < s$, and $y_k \neq y_{k+1}$ for any k with $r \leq k < s$. Then (4.7) and (4.4) together yield $d_r = \cdots = d_{s-1} = 2$, and $\{y_r, \dots, y_s\} \subseteq D$. Then also (4.8) yields

$$1/m_k + 1/m_{k+1} + 1/m(h) \leq 1 \qquad (**)$$

for all k with $r \leq k < s$. In particular, we have $m_k + m_{k+1} > 4$ for all such k.

Suppose $s - r \geq 3$, and take $k = r + 1$, so that $\{k-1, \dots, k+2\} \subseteq \mathscr{S}_4$, and we are in one of the two situations described in (4.6). Suppose (4.6)(i) applies. Thus, $k \in \mathscr{S}_0$, $\{k \pm 1\} \subseteq \mathscr{S}_1$, (h_{k-2}) and (h_{k+2}) are basic special segments of type I, and $m_{k-1} = m_{k+1} = 3$. Consulting (3.6) we see that $d_{k-2} = d_{k+2} = 4$, and $m_{k-2} = m_{k+2} = 2$. Apply $(**)$ with $k-2$ in place of k, and conclude that $m(h) \geq 6$. Then $\frac{1}{2}a(h) \geq 12$, and this implies $l(I \cap J_{k-2}) \geq 3$ or $l(I \cap J_{k+2}) \geq 3$, which is contrary to (4.7).

Now (4.6)(ii) must hold, so that $d_{k-2} = d_{k+2} = 6$, $m_{k-1} = m_{k+1} = 2$, and $m_{k-2} = m_k = m_{k+2} = 3$. Again, $(**)$ yields $m(h) \geq 6$ and $l(I) \geq 12$, with a contradiction as in the preceding paragraph. We therefore conclude that $s - r \leq 2$.

Suppose $s - r = 2$. Then $d(y_r, y_s) \leq 4$ and $l(I) \leq 8$, so that $m(h) \leq 4$. Then $(**)$ yields $m_r + m_{r+1} > 5$. We may appeal to (4.5), with $k = r$, where we see that the condition $m_r + m_{r+1} > 5$ immediately rules out all but (4.5)(iii). But in that case we have $m_{r-1} = 2$, and applying $(**)$, with $r - 1$ in place of k, again yields a contradiction. We conclude that $s = r + 1$.

Now $d(y_r, y_s) = 2$ and $l(I) \geq 6$, so that $a(h) = 8$ or 12. Suppose $a(h) = 8$. Then $(**)$ yields $m_k > 2$ for $k \in \{r-1, r, r+1\}$. But $d_r = 2$, so (4.4) provides a contradiction. Thus, $a(h) = 12$ and hence $l(I) = 6$. Further, (4.7) yields $\{u, y_r, y_{r+1}, v\} \subseteq D$ and $d(u, y_r) = d(y_r, y_{r+1}) = d(y_{r+1}, v) = 2$. Since $(**)$ holds for both $k = r$ and $k = r + 1$, we may exclude the cases given by parts (ii), (iv), and (v) of (4.4), with $k = r$. Thus (4.4)(i) or (iii) applies, and this yields outcome (iii) of the lemma, as desired.

5. CONTROL OF CONVERGENCE

We now show that the polarization $\mathscr{P}$, defined in (3.12), controls convergence relative to $(b, \mathscr{D})$.

We continue with all of the notation introduced in the preceding section. In particular, we have $J \in \mathscr{P}(\mathbf{h}')$, $\mathbf{h}$ a distillation of a braiding $\mathbf{h}''$ of $\mathbf{h}'$, and Σ a simplex over J such that $\mathbf{h}$ and Σ are married. Further, we have $h \in \mathscr{E}(\mathbf{h}', J)$.

5.1. LEMMA. *$\mathscr{P}$ satisfies the "Exclusion" condition* (2.8)(1).

Proof. Suppose $\mathbf{h}' \neq \emptyset$. Then (4.3) and (4.7) together imply that $J_0 \not\subseteq A(h)$. Recall the notation: $I = J \cap A(h)$, and notice that an immediate consequence of (4.9) is that $l(I) \leq \frac{1}{2}a(h)$. Finally, suppose $\partial J \cap I \neq \emptyset$. Then (4.3) and (4.7) yield $l(I) \leq 2$. Since $b(h) \geq 8$, by (1.12), we then have $b(h) > 2 \cdot l(I)$, and all pats of (2.8)(1) have been verified.

The problem now is to verify that $\mathscr{P}$ satisfies the "Extension" condition (2.8)(2). To do this, we first handle the cases where $\partial J \cap I \neq \emptyset$. In particular, we need to examine the case where J is degenerate.

5.2. LEMMA. *Let K be a segment of $A(h)$. Assume that $l(K) \geq \frac{1}{2}b(h)$, with strict inequality if $\partial K \subseteq D$. Then $K \in \mathscr{P}((h))$.*

Proof. Notice that (h) is the unique distillation of the unique braiding of (h), so it suffices to show that (h) and K are married. In terms of the decomposition (4.1) we take $(h) = \mathbf{a}_1$ and we obtain $l(K \cap A(h)) \geq 2 \cdot m(h)$ (with strict inequality if $\partial K \subseteq D$), since $m(h) = \frac{1}{2}b(h)$. Thus (h) is an ordinary segment of (h), relative to K, in the sense of definition (3.10). Then (h) and K are married, and thus $K \in \mathscr{P}((h))$.

5.3. LEMMA. *Suppose that we have a point $w \in A(h)$, with $x \in [w, w \cdot h)$. Suppose further that $d(x, w \cdot h) \geq 2 \cdot m(h)$, and that strict inequality holds if $\{x, w\} \subseteq D$. Put $\tilde{J} = [w \cdot h, x']$. Then $\tilde{J} \in \mathscr{P}(\mathbf{h}' \circ (h))$.*

Proof. By assumption we have $x \in I$. If $J = \{x\}$ then $\mathbf{h} = (\varnothing)$ by (4.3), and then $\tilde{J} \in \mathscr{P}((h))$ by (5.2). Thus, we may assume that J is non-degenerate, whence also $\mathbf{h} \neq (\varnothing)$. Now (4.3) and (4.7) together show that $I = [x, v] \leq J_0$. Moreover, (4.7) yields $d(x, v) \leq 2$, with strict inequality unless $\{x, v\} \subseteq D$.

Put $\tilde{\mathbf{h}} = (h) \circ \mathbf{h}$, and notice that $\tilde{\mathbf{h}}$ is a simple distillation of $\mathbf{h} \circ (h)$, of codegree 1, as defined in (2.1). Hence $\tilde{\mathbf{h}}$ is a distillation of $\mathbf{h}'' \circ (h)$, which is in turn a braiding of $\mathbf{h}' \circ (h)$. Next, define points $\tilde{x}_0, \ldots, \tilde{x}_{n+2}$ by

$$\tilde{x}_0 = w \cdot h, \qquad \tilde{x}_i = x_{i-1}\,(1 \leq i \leq n+2).$$

Put $\tilde{\Sigma} = \Sigma(\tilde{x}_0, \ldots, \tilde{x}_{n+2})$. Also, put

$$\tilde{y}_1 = v, \qquad \tilde{y}_i = y_{i-1}\,(2 \leq i \leq n+1).$$

Then $\tilde{\Sigma}$ is a simplex over $\tilde{J}$, relative to the partition $(\tilde{x}_0 < \tilde{y}_1 \leq \cdots \leq \tilde{y}_{n+1} < \tilde{x}_{n+2})$. To see this, we need only sketch a picture, as shown in Fig. 5. Moreover, Fig. 5 displays that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are engaged. In order to complete the proof of (5.3), it now suffices to show that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married.

Write $\mathbf{h} = \mathbf{a}_1 \circ \mathbf{x}_1 \circ \cdots \circ \mathbf{a}_m \circ \mathbf{x}_m \circ \mathbf{a}_{m+1}$ as in (4.1). Put $\tilde{\mathbf{a}}_1 = (h) \circ \mathbf{a}_1$, $\tilde{\mathbf{x}}_i = \mathbf{x}_i$ for all i, and $\tilde{\mathbf{a}}_j = \mathbf{a}_j$ for $j > 1$. It is immediate that each $\tilde{\mathbf{x}}_i$ is

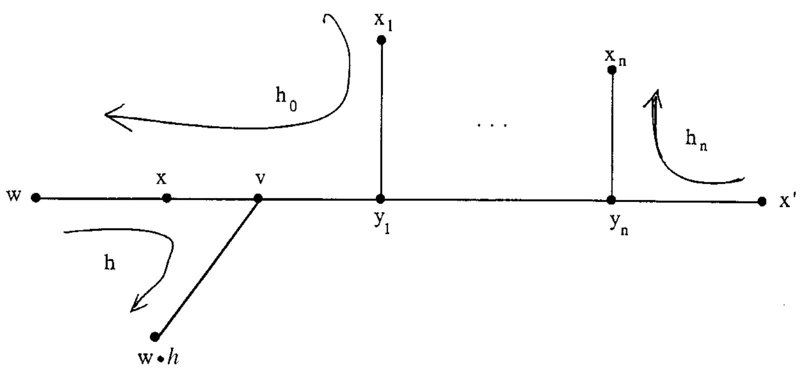


FIGURE 5

extraspecial and that each $\tilde{a}_j$ ($j > 1$) is ordinary, relative to $\tilde{\Sigma}$. But also, we have $d(\tilde{x}_0, \tilde{x}_1) = d(w \cdot h, x) \geq 2 \cdot m(h)$, with strict inequality if $\{x, w\} \subseteq D$. Here $w \in D$ if and only if $w \cdot h \in D$, since $a(h)$ is even. Thus $l(\tilde{\Sigma} \cap A(h)) \geq 2 \cdot m(h)$, with strict inequality if $(\tilde{\Sigma} \cap A(h)) \subseteq D$. This shows that $\tilde{\mathbf{a}}_1$ is ordinary, and so $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married.

Remark. With (5.3), we have shown that part (b) of the Extension condition (2.8)(2) is satisfied by $\mathscr{P}$.

In order to complete the proof that $\mathscr{P}$ controls convergence relative to $(b, \mathscr{D})$, it only remains to deal with part (a) of the Extension condition (2.8)(2). Thus, we may now assume:

5.4. We have $\partial J \cap A(h) = \emptyset$, and $l(I) \geq a(h) - 2 \cdot m(h)$, with strict inequality if $\partial I \not\subseteq D$.

Our goal now will be to show that $[x \cdot h, x'] \in \mathscr{P}(\mathbf{h}' \circ (h))$.

Recall the notation from Section 4: we have $I = [u, v] \leq [x, x']$, and we have $\mathscr{S}_4 = \{r, r+1, \ldots, s\}$, defined by $k \in \mathscr{S}_4$ if $y_k \in I - \{u, v\}$. Put

$$\tilde{\mathbf{h}} = (h_0, \ldots, h_{r-1})^h \circ (h) \circ (h_s, \ldots, h_n),$$

and notice that $\tilde{\mathbf{h}}$ is a simple distillation of $\mathbf{h} \circ (h)$, of codegree 1. Then $\tilde{\mathbf{h}}$ is a distillation of $\mathbf{h}'' \circ (h)$, which is itself a braiding of $\mathbf{h}' \circ (h)$. (See (2.1) and (2.2).) Put

$$\tilde{J} = [x \cdot h, x'].$$

We will show that there exists a suitable simplex $\tilde{\Sigma}$ over $\tilde{J}$, such that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married.

There are three cases to consider, corresponding to the three possible outcomes in (4.9). We will treat each case separately.

5.5. Lemma. *Suppose* $\mathscr{S}_4 = \{r\}$, $a(h) = 8$, *and* $d(u, y_r) = d(y_r, v) = 2$. *Then* $\tilde{J} \in \mathscr{P}(\mathbf{h}' \circ (h))$.

Proof. Define points $\tilde{x}_i$, $0 \leq i \leq n+2$, as

$$\tilde{x}_0 = x_0 \cdot h, \ldots, \tilde{x}_{r-1} = x_{r-1} \cdot h,$$

$$\tilde{x}_r = y_r \cdot h, \qquad \tilde{x}_{r+1} = y_r,$$

$$\tilde{x}_{r+2} = x_{r+1}, \ldots, \tilde{x}_{n+2} = x_{n+1}.$$

Put $\tilde{\Sigma} = \Sigma(\tilde{x}_0, \ldots, \tilde{x}_{n+2})$. Also define points $\tilde{y}_i$, $1 \leq i \leq n+1$, as

$$\tilde{y}_1 = y_1 \cdot h, \ldots, \tilde{y}_{r-1} = y_{r-1} \cdot h,$$

$$\tilde{y}_r = u \cdot h, \qquad \tilde{y}_{r+1} = v,$$

$$\tilde{y}_{r+2} = y_{r+1}, \ldots, \tilde{y}_{n+1} = y_n.$$

We have the picture of Σ shown in Fig. 6.

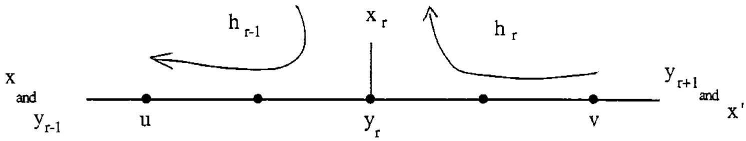


FIGURE 6

Since $a(h) = 8$, and since $[u, v]$ is oriented toward v by h (in accordance with (2.7)(c)), we then have the picture of $\tilde{\Sigma}$ shown in Fig. 7.

Evidently, $\tilde{\Sigma}$ is a simplex over $\tilde{J}$, relative to the partition $(\tilde{x}_0 < \tilde{y}_1 \leq \cdots \leq \tilde{y}_{n+1} < \tilde{x}_{n+2})$ of $\tilde{J}$. Further, it is evident that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are engaged. It remains to show that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married.

Suppose $\{r-1, r\} \subseteq \mathscr{S}_0$. In terms of the decomposition (4.1), this means that (h_{r-1}, h_r) is part of an ordinary segment $\mathbf{a}_k$ of $\mathbf{h}$, $1 \leq k \leq m+1$. Put $\mathbf{x} = (h_{r-1}^h, h, h_r)$. We claim that $\mathbf{x}$ is an extraspecial segment of $\tilde{\mathbf{h}}$, relative to $\tilde{\Sigma}$. Indeed, first notice that (h) is a special segment of type I, as is revealed by a glance at the definition (3.6)(I). Then notice that (4.8) yields

$$1/m\left(h_{r-1}^h\right) + 1/m(h_r) \leq 1/2,$$

so that $\mathbf{x}$ satisfies (3.9)(b). Since $r - 1 \in \mathscr{S}_0$, (3.10) gives $d(x_{r-1}, x_r) \geq 2 \cdot m_r$ with strict inequality if $\{x_{r-1}, x_r\} \subseteq D$. Also $d(y_r, x_r) \leq 2$, with strict inequality unless $\{x_r, y_r\} \subseteq D$, by (3.5). But $l(\tilde{\Sigma} \cap A(h_{r-1}^h)) \geq d(x_{r-1} \cdot h, y_r \cdot h) = d(x_{r-1}, y_r)$, so we get

$$l\left(\tilde{\Sigma} \cap A\left(h_{r-1}^h\right)\right) \geq 2 \cdot m_{r-1} - 2,$$

with strict inequality if $\partial(\tilde{\Sigma} \cap A(h_{r-1}^h)) \subseteq D$. Since $p((h)) = 2$, according to (3.8), we now see that h_{r-1}^h satisfies (3.9)(c) with respect to $\mathbf{x}$ and $\tilde{\Sigma}$. A symmetric argument yields also (3.9)(c) for h_r. Thus, $\mathbf{x}$ is extraspecial.

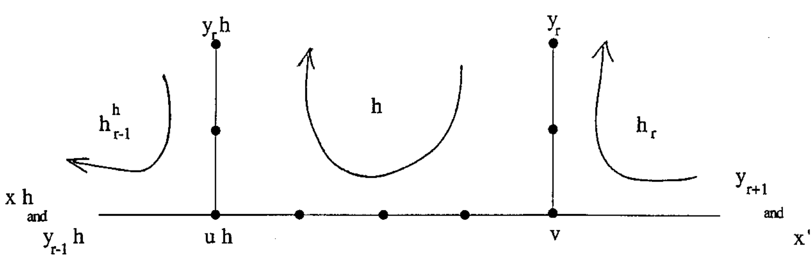


FIGURE 7

Now write $\mathbf{a}_k = \mathbf{b}_k \circ (h_{r-1}, h_r) \circ \mathbf{c}_k$. We then have

$$\tilde{\mathbf{h}} = \mathbf{a}_1^h \circ \mathbf{x}_1^h \circ \cdots \circ \mathbf{x}_{k-1}^h \circ \mathbf{b}_k^h \circ \mathbf{x} \circ \mathbf{c}_k \circ \mathbf{x}_k \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1},$$

and this alternating sequence of ordinary and extraspecial segments shows that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married in this case.

Suppose next that $r-1 \in \mathscr{S}_0$ and $r \in \mathscr{S}_1$. Thus, (h_{r-1}) comes at the end of an ordinary segment $\mathbf{a}_k$, and (h_r) comes at the beginning of a basic extraspecial segment $\mathbf{x}_k$ of $\mathbf{h}$. Put $\mathbf{x} = (h_{r-1}^h, h) \circ \mathbf{x}_k$. We will now show that $\mathbf{x}$ is an extraspecial segment of $\tilde{\mathbf{h}}$, relative to $\tilde{\Sigma}$. Indeed, the proof is almost the same as in the preceding case. Thus, we observe that (h) is a special segment of type I, and $1/m(h_{r-1}^h) + 1/m(h_r) \leq \frac{1}{2}$, so that $\mathbf{x}$ satisfies (3.9)(b). With (h) of type I we have $p((h)) = q((h)) = 2$, by (3.8). Let $\mathbf{s}$ be the basic special segment which follows (h_r) in $\mathbf{x}_k$. Applying (3.9)(c) to $\mathbf{x}_k$, we have $d(x_r, x_{r+1}) \geq 2 \cdot m_r - p(\mathbf{s})$, with strict inequality if $\{x_r, x_{r+1}\} \subseteq D$. Now $d(\tilde{x}_{r+1}, \tilde{x}_{r+2}) = d(y_r, x_{r+1}) \geq d(x_r, x_{r+1}) - 2$, with strict inequality unless $\{x_r, y_r\} \subseteq D$, by (3.5). Thus, we get

$$l\big(\tilde{\Sigma} \cap A(h_r)\big) \geq 2 \cdot m_r - p(\mathbf{s}) - q((h)),$$

with strict inequality if $\{y_r, x_{r+1}\} \subseteq D$. This is half of what we need in order to show that $\mathbf{x}$ satisfies (3.9)(c). The other half is

$$l\big(\tilde{\Sigma} \cap A\big(h_{r-1}^h\big)\big) \geq 2 \cdot m_{r-1} - p((h)),$$

which is proved by exactly the same argument as in the case $\{r-1, r\} \subseteq \mathscr{S}_0$, above. Thus, $\mathbf{x}$ is extraspecial relative to $\tilde{\Sigma}$.

Now write $\mathbf{a}_k = \mathbf{b}_k \circ (h_{r-1})$, and obtain

$$\tilde{\mathbf{h}} = \mathbf{a}_1^h \circ \mathbf{x}_1^h \circ \cdots \circ \mathbf{x}_{k-1}^h \circ \mathbf{b}_k^h \circ \mathbf{x} \circ \mathbf{a}_{k+1} \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1}.$$

This alternating sequence of ordinary and extraspecial segments shows that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married in this case. Almost the same argument handles the case where $r-1 \in \mathscr{S}_1$ and $r \in \mathscr{S}_0$, so we may assume henceforth that neither $r-1$ nor r is in $\mathscr{S}_0$.

Suppose next that $\{r-1, r\} \subseteq \mathscr{S}_1$. Thus, there are basic extraspecial segments $\mathbf{x}_{k-1}$ and $\mathbf{x}_k$ of $\mathbf{h}$ such that $\mathbf{x}_{k-1}$ ends in (h_{r-1}), $\mathbf{x}_k$ begins in (h_r), and the ordinary segment $\mathbf{a}_k$ is empty. In this case we put $\mathbf{x} = \mathbf{x}_{k-1}^h \circ (h) \circ \mathbf{x}_k$ and we show that $\mathbf{x}$ is extraspecial relative to $\tilde{\Sigma}$. Let $\mathbf{s}$ be the basic special segment which follows (h_r) in $\mathbf{x}_k$, and let $\mathbf{t}$ be the basic special segment which precedes (h_{r-1}) in $\mathbf{x}_{k-1}$. Again, (h) is special of type I, and $\mathbf{x}$ satisfies (3.9)(b), so we need to show

$$l\big(\tilde{\Sigma} \cap A\big(h_{r-1}^h\big)\big) \geq 2 \cdot m_{r-1} - p((h)) - q(\mathbf{t}),$$

$$l\big(\tilde{\Sigma} \cap A(h_r)\big) \geq 2 \cdot m_r - p(\mathbf{s}) - q((h)),$$

and with the familiar requirements concerning strict inequality. But the argument for $\tilde{\Sigma} \cap A(h_r)$ has already been provided in the case $r - 1 \in \mathscr{S}_0$, $r \in \mathscr{S}_1$, considered above. The argument for $\tilde{\Sigma} \cap A(h_{r-1}^h)$ may then safely be deleted, since it is essentially the same as that for $\tilde{\Sigma} \cap A(h_r)$. Thus, $\mathbf{x}$ is extraspecial relative to $\tilde{\Sigma}$, and then $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married, as is seen from the decomposition

$$\tilde{\mathbf{h}} = \mathbf{a}_1^h \circ \mathbf{x}_1^h \circ \cdots \circ \mathbf{a}_{k-1}^h \circ \mathbf{x} \circ a_{k+1} \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1}.$$

If $\{r-1, r\} \subseteq \mathscr{S}_3$ then (3.6) shows that $m_{r-1} = 2$ or $m_r = 2$. But $m(h) = 2$, so we contradict (4.8) in this case. By symmetry, we may therefore assume henceforth that $r - 1 \in \mathscr{S}_1 \cup \mathscr{S}_2$ and $r \in \mathscr{S}_3$. Thus, there is a basic extraspecial segment $\mathbf{x}$ of $\mathbf{h}$ which, in the notation of (4.2), satisfies $(h_{r-1}) = (h_{k_i})$ for some i, $1 \le i \le u$. Write $\mathbf{x} = \mathbf{y} \circ \mathbf{s}_i \circ \mathbf{z}$, where $\mathbf{s}_i$ is the basic special segment that begins with (h_r). Then put $\tilde{\mathbf{s}} = (h) \circ \mathbf{s}_i$ and put $\tilde{\mathbf{x}} = \mathbf{y}^h \circ \tilde{\mathbf{s}} \circ \mathbf{z}$. We will now show that $\tilde{\mathbf{s}}$ is a special segment, and $\tilde{\mathbf{x}}$ an extraspecial segment, of $\tilde{\mathbf{h}}$ relative to $\tilde{\Sigma}$.

To see that $\tilde{\mathbf{s}}$ is special, notice first of all that since $d(y_r \cdot h, u \cdot h) = d(y_r, v) = 2$, $\tilde{\mathbf{s}}$ satisfies the condition $(*)$ preceding (3.6). Next, since $r \in \mathscr{S}_3$ we have $m_r = 2$ or 3, and then (4.8) forces $m_r = 3$ and $m_{r-1} \ge 6$. With $m_r = 3$ we conclude that $\mathbf{s}_i$ is special of type II, and $m(\tilde{s})$ is an alternating string of 2's and 3's.

Notice that $l(\tilde{J} \cap A(h_r)) = d_r - 2$, since $[y_r, v]$ is now part of $\tilde{\Sigma}$ but not of $\tilde{J}$. (See the pictures above.) Suppose $l(\mathbf{s}_i) = 1$. Then (3.6)(II) gives $d_r = 6$, so with respect to $\tilde{\Sigma}$ we get $m((h, h_r)) = (2, 3)$ and $d((h, h_r)) = (4, 4)$, as in (3.6)(II)(ii). Thus, $\tilde{\mathbf{s}}$ is special of type II in this case. Similarly, if $l(\mathbf{s}_i) = 2$ then $m(\mathbf{s}_i) = (3, 2)$ and $d(\mathbf{s}_i) = (4, 4)$, and with respect to $\tilde{\Sigma}$ we get $m(\tilde{\mathbf{s}}) = (2, 3, 2)$ and $d(\tilde{\mathbf{s}}) = (4, 2, 4)$. Again, $\tilde{\mathbf{s}}$ is special of type II. According to (3.6)(II), it remains to consider the case where $l(\mathbf{s}_i) > 4$, and where $d(\mathbf{s}_i) = (4, 2) \circ \xi_t \circ \mu$, with $t = [l(\mathbf{s}_i)/4]$. By the remark following (3.6), $l(\mathbf{s}_i) \not\equiv 3 \pmod 4$, so that $t = [l(\tilde{\mathbf{s}})/4]$. Now $d(\tilde{\mathbf{s}}) = (4, 2, 2) \circ \xi_t \circ \mu$, and thus $\tilde{\mathbf{s}}$ is special of type II.

We next show that $\tilde{\mathbf{x}}$ is extraspecial. We have the decomposition

$$\tilde{\mathbf{x}} = \big((h_{k_1}) \circ \mathbf{s}_1 \circ \cdots \circ \mathbf{s}_{i-1} \circ (h_{k_1})\big)^h \circ \tilde{\mathbf{s}} \circ (h_{k_{i+1}}) \circ \cdots \circ \mathbf{s}_{k_u} \circ (h_{k_{u+1}}).$$

This is a chain of special sequences relative to $\tilde{\Sigma}$, together with "connecting elements" for which we must verify the conditions in (3.9). Notice that (3.9)(b) holds since $\mathbf{s}_i$ is of type II, forcing $m(h_{k_i}) \ge 6$ and $m(h_{k_{i+1}}) \ge 6$. It only remains to show that

$$l\big(\tilde{\Sigma} \cap A(h_{r-1}^h)\big) \ge 2 \cdot m_{r-1} - p(\tilde{\mathbf{s}}) - q(\mathbf{s}_{i-1}^h),$$

with strict inequality if $\{x_{r-1} \cdot h, y_r \cdot h\} \subseteq D$, and where we take $q(\mathbf{s}_{i-1}^h) = 0$ if $i = 1$. Here, (3.8) gives $p(\tilde{\mathbf{s}}) = 4$ since $\tilde{\mathbf{s}}$ is of type II, and since $\tilde{\mathbf{s}}$ begins with (h), where $m(h) = 2$. On the other hand, $p(\mathbf{s}_i) = 2$, also by (3.8). Now $q(\mathbf{s}_{i-1}^h) = q(\mathbf{s}_{i-1})$, so if we apply (3.9)(c) to $\mathbf{x}$ then we get

$$l(\Sigma \cap A(h_{r-1})) \geq 2 \cdot m_{r-1} - p(\tilde{\mathbf{s}}) - q(\mathbf{s}_{i-1}^h) + 2,$$

with strict inequality if $\{x_{r-1}, x_r\} \subseteq D$. But $l(\tilde{\Sigma} \cap A(h_{r-1}^h)) = l(\Sigma \cap A(h_{r-1})) - d(y_r, x_r)$. Since $d(y_r, x_r) \leq 2$, with equality only if $\{x_r, y_r\} \subseteq D$, we have the desired result for $\tilde{\Sigma} \cap A(h_{r-1}^h)$.

Now $\tilde{\mathbf{x}}$ is extraspecial. Taking $\mathbf{x}$ to be $\mathbf{x}_k$ in the decomposition (4.1), we have

$$\tilde{\mathbf{h}} = \mathbf{a}_1^h \circ \mathbf{x}_1^h \circ \cdots \circ \mathbf{a}_k^h \circ \tilde{\mathbf{x}} \circ \mathbf{a}_{k+1} \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1},$$

and evidently $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married. This completes the proof of (5.5).

5.6. LEMMA. *Suppose* $\mathcal{S}_4 = \{r, r+1\}$, $y_r = y_{r+1}$, $a(h) = 8$, *and* $d(u, y_r) = d(y_{r+1}, v) = 2$. *Then* $\tilde{J} \in \mathcal{P}(\mathbf{h}' \circ (h))$.

Proof. Define points $\tilde{x}_i$, $0 \leq i \leq n+1$, as

$$\tilde{x}_0 = x_0 \cdot h, \ldots, \tilde{x}_{r-1} = x_{r-1} \cdot h,$$
$$\tilde{x}_r = y_r \cdot h, \qquad \tilde{x}_{r+1} = y_r = y_{r+1},$$
$$\tilde{x}_{r+2} = x_{r+2}, \ldots, \tilde{x}_{n+1} = x_{n+1}.$$

Put $\tilde{\Sigma} = \Sigma(\tilde{x}_0, \ldots, \tilde{x}_{n+1})$. Also, define points $\tilde{y}_j$, $1 \leq j \leq n$, as

$$\tilde{y}_1 = y_1 \cdot h, \ldots, \tilde{y}_{r-1} = y_{r-1} \cdot h,$$
$$\tilde{y}_r = u \cdot h, \qquad \tilde{y}_{r+1} = v,$$
$$\tilde{y}_{r+2} = y_{r+2}, \ldots, \tilde{y}_n = y_n.$$

We have the picture of Σ shown in Fig. 8. In Fig. 8 we are using (4.4) which tells us that, since $d_r = 0$, (h_{k-1}) and (h_{k+1}) are basic special segments of type I. Since $a(h) = 8$ and $[u, v]$ is oriented toward v by h, we then have the picture of $\tilde{\Sigma}$ shown in Fig. 9. Figure 9 shows that $\tilde{\Sigma}$ is a simplex over $\tilde{J}$, relative to the partition $(\tilde{x}_0 < \tilde{y}_1 \leq \cdots \leq \tilde{y}_n < \tilde{x}_{n+1})$ of $\tilde{J}$, and that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are engaged.

We note that (4.4) says that $r \in \mathcal{S}_2$, so there is a basic extraspecial segment $\mathbf{x}$ of $\mathbf{h}$, such that in the notation of (4.2), we have

$$(h_{r-1}) = \mathbf{s}_{i-1}, \qquad h_r = h_{k_1}, \qquad (h_{r+1}) = \mathbf{s}_i,$$

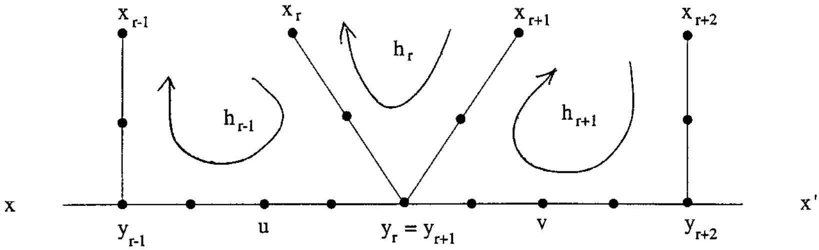


FIGURE 8

for some i with $1 < i \leq u$. Put $\tilde{\mathbf{s}} = (h_{r-1}^{h}, h, h_{r+1})$ and put

$$\tilde{\mathbf{x}} = \left(\left(h_{k_1}\right) \circ \mathbf{s}_1 \circ \cdots \circ \left(h_{k_{i-1}}\right)\right)^{h} \circ \tilde{\mathbf{s}} \circ \left(h_{k_{i+1}}\right) \circ \cdots \circ s_{k_u} \circ \left(h_{k_{u+1}}\right).$$

One sees from the picture of $\tilde{\Sigma}$ that $\tilde{\mathbf{s}}$ is special of type I, relative to $\tilde{\Sigma}$. (See (3.6) for this.) Also (4.4) says that $m_k = 3$, so (3.9)(b) applied to $\mathbf{x}$ yields $m_{k_{i-1}} \geq 6$ and $m_{k_{i+1}} \geq 6$. Thus, $\tilde{\mathbf{x}}$ satisfies (3.9)(b). The other parts of (3.9) pass unchanged from $\mathbf{x}$ to $\tilde{\mathbf{x}}$, so $\tilde{\mathbf{x}}$ is extraspecial relative to $\tilde{\Sigma}$. Take $\mathbf{x} = \mathbf{x}_k$ in the decomposition (4.1). We then get

$$\tilde{\mathbf{h}} = (\mathbf{a}_1 \circ \mathbf{x}_1 \circ \cdots \circ \mathbf{x}_{k-1} \circ \mathbf{a}_k)^{h} \circ \tilde{\mathbf{x}} \circ a_{\mathbf{k}+1} \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1},$$

and this decomposition shows that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married. This proves (5.6).

5.7. Lemma. *Suppose* $\mathscr{S}_4 = \{r, r+1\}$, $y_r \neq y_{r+1}$, $a(h) = 12$, *and* $d(u, y_r) = d(y_r, y_{r+1}) = d(y_{r+1}, v) = 2$. *Then* $\tilde{J} \in \mathscr{P}(\mathbf{h}' \circ (h))$.

Proof. Define the points $\tilde{x}_i$, $0 \leq i \leq n+1$, the simplex $\tilde{\Sigma}$, and the points $\tilde{y}_j$, $1 \leq j \leq n$, exactly as in the proof of (5.6).

By (4.9), we have two cases to consider here, one in which $r \in \mathscr{S}_0$ and $m_r = 2$, and the other where $r \in \mathscr{S}_2$ and $m_r = 6$. In the second of these cases we get further information from (4.4)(iii). In particular, $d_{k-1} = d_{k+1} = 4$ if $r \in \mathscr{S}_2$. If $r \in \mathscr{S}_0$ then since $m_r = 2$ and $m(h) = 3$, (4.8) yields

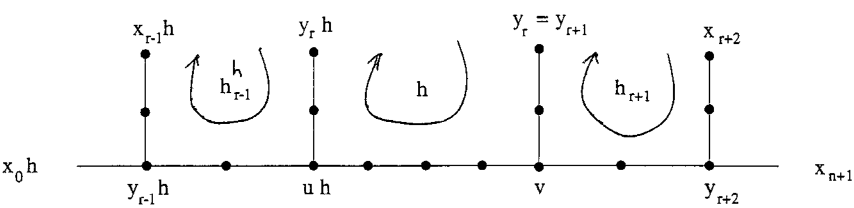


FIGURE 9

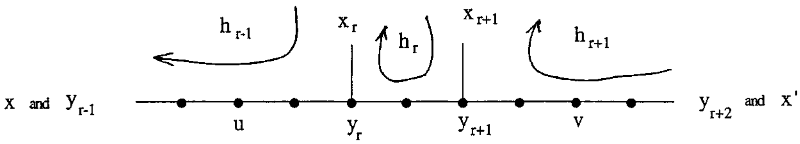


FIGURE 10

$m_{r-1} \geq 6$ and $m_{r+1} \geq 6$. Here $\{r-1, r+1\} \subseteq \mathscr{S}_0 \cup \mathscr{S}_1$, so it follows from (3.9)(c) and (3.10) that $d_{r-1} \geq 4$ and $d_{r+1} \geq 4$. Thus, in any case we have the picture of σ shown in Fig. 10.

Since $a(h) = 12$, this yields the picture of $\tilde{\tilde{\Sigma}}$ shown in Fig. 11.

Suppose first that $r \in \mathscr{S}_0$. This leads to the consideration of four cases, according to whether $r \pm 1 \in \mathscr{S}_0$, $r - 1 \in \mathscr{S}_0$ and $r + 1 \in \mathscr{S}_1$, $r - 1 \in \mathscr{S}_1$, and $r + 1 \in \mathscr{S}_0$, and the case $r \pm 1 \in \mathscr{S}_1$. In any case, the above picture shows that (h) is a special segment of $\tilde{\mathbf{h}}$, of type II, relative to $\tilde{\tilde{\Sigma}}$. Write $\mathbf{h} = \mathbf{y} \circ \mathbf{b} \circ (h_r) \circ \mathbf{c} \circ \mathbf{z}$, where $\mathbf{b}$ is a basic extraspecial segment if $r - 1 \in \mathscr{S}_1$, and $\mathbf{b} = (h_{r-1})$ if $r - 1 \in \mathscr{S}_0$, and where $\mathbf{c}$ is defined similarly, relative to $r + 1$. Put $\tilde{\mathbf{x}} = \mathbf{b}^h \circ (h) \circ \mathbf{c}$.

We claim that $\tilde{\mathbf{x}}$ is an extraspecial segment of $\tilde{\mathbf{h}}$, relative to $\tilde{\tilde{\Sigma}}$. Since $m_{r \pm 1} \geq 6$, as we have seen, we need only verify (3.9)(c) for $\tilde{\mathbf{x}}$. By symmetry, it will suffice to show

$$l\big(\tilde{\tilde{\Sigma}} \cap A(h_{r+1})\big) \geq 2 \cdot m_{r+1} - p - q((h)),$$

where $p = 0$ if $r + 1 \in \mathscr{S}_0$, where $p = p(\mathbf{s})$ if $r + 1 \in \mathscr{S}_1$ and $\mathbf{s}$ is the basic special segment following (h_{r+1}), and that the above inequality is strict if $\partial(\tilde{\tilde{\Sigma}} \cap A(h_{r+1})) \subseteq D$. The argument for this is perhaps familiar from the proof of (5.6), but we will supply it again.

We have $l(\tilde{\tilde{\Sigma}} \cap A(h_{r+1})) = d(y_{r+1}, x_{r+2}) = d(x_{r+1}, x_{r+2}) - d(x_{r+1}, y_{r+1})$. Here (3.9)(c) and (3.10) yield

$$(x_{r+1}, x_{r+2}) \geq 2 \cdot m_{r+1} - p,$$

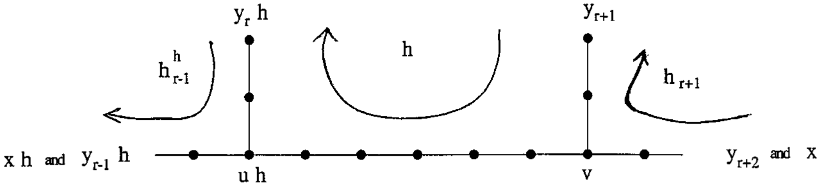


FIGURE 11

with strict inequality if $\{x_{r+1}, x_{r+2}\} \subseteq D$, and (3.5) yields $d(x_{r+1}, y_{r+1}) \leq 2$, with strict inequality unless $\{x_{r+1}, y_{r+1}\} \subseteq D$. Now simply observe that $q((h)) = 2$ since $m(h) = 3$, in accordance with (3.8). This yields

$$d(y_{r+1}, x_{r+2}) \geq 2 \cdot m_{r+1} - p - q((h))$$

with the required condition concerning strict inequality, as desired. Thus, $\tilde{\mathbf{x}}$ is extraspecial relative to $\tilde{\Sigma}$. As in the proofs of (5.5) and (5.6), it now follows that $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married. We are therefore finished with the case $r \in \mathscr{S}_0$.

We now have the case $r \in \mathscr{S}_2$ and $m_r = 6$. Let $\mathbf{x}$ be the basic extraspecial segment of $\mathbf{h}$ involving (h_r). Then let $\mathbf{s}$ and $\mathbf{t}$ be the basic special segments preceding and, respectively, following (h_r) in $\mathbf{x}$. Put $\tilde{\mathbf{s}} = \mathbf{s}^h \circ (h) \circ \mathbf{t}$. By (4.4), $\mathbf{s}$ and $\mathbf{t}$ are of type II and $m_{k-1} = m_{k+1} = 2$. Then (3.6) shows that $l(\mathbf{s}) \equiv 2$ or $3 \pmod 4$, and similarly for $l(\mathbf{t})$.

We claim that $\tilde{\mathbf{s}}$ is a special segment of $\tilde{\mathbf{h}}$, of type II, relative to $\tilde{\Sigma}$. Since $m(h) = 3$, $m(\tilde{\mathbf{s}})$ is indeed an alternating string of 2's and 3's, and it follows from our picture of $\tilde{\Sigma}$ that $\tilde{\mathbf{s}}$ satisfies the condition $(*)$ preceding (3.6). Our picture shows that, with respect to $\tilde{\Sigma}$, we have $d(h^h_{r-1}) = d_{r-1} - 2$ and $d(h_{r+1}) = d_{r+1} - 2$, and for any other entries in $\mathbf{s}$ or $\mathbf{t}$, the value of the "d-function" is unchanged on passage from Σ to $\tilde{\Sigma}$. Further, $d(h) = 6$ relative to $\tilde{\Sigma}$.

Suppose $l(\mathbf{s}) = l(\mathbf{t}) = 2$. Then $m(\tilde{\mathbf{s}}) = (3, 2, 3, 2, 3)$ and $d(\tilde{\mathbf{s}}) = (4, 2) \circ (6) \circ (2, 4)$, as required for (3.6)(II)(iv). Thus, $\tilde{\mathbf{s}}$ is special in this case. If $l(\mathbf{s}) = 2$ and $l(\mathbf{t}) = 3$ then $m(\tilde{\mathbf{s}}) = (3, 2, 3, 2, 3, 2)$ and $d(\tilde{\mathbf{s}}) = (4, 2) \circ (6) \circ (2, 2, 4)$, and again $\tilde{\mathbf{s}}$ is special. Now suppose $l(\mathbf{s}) = 2$ and $l(\mathbf{t}) > 4$. We then have $d(\mathbf{t}) = (4, 2, 2) \circ \xi \circ \mu$ where $\xi = \xi_t$, $t = [l(\mathbf{t})/4]$, and where $\mu = (2, 4)$ if $m(\mathbf{t})$ ends in 3, and $\mu = (2, 2, 4)$ if $m(\mathbf{t})$ ends in 2. From this we get

$$d(\tilde{\mathbf{s}}) = (4, 2) \circ (6) \circ (2, 2, 2) \circ \xi \circ \mu.$$

Since $l(\mathbf{t}) \not\equiv 1 \pmod 4$, $[l(\tilde{\mathbf{s}})/4] = t + 1$, and $d(\tilde{\mathbf{s}}) = (4, 2) \circ \xi_{t+1} \circ \mu$, as required. The reader may now check that we also get $\tilde{\mathbf{s}}$ special of type II if $l(\mathbf{s}) = 3$. By symmetry, we may then assume that $l(\mathbf{s}) > 4$ and $l(\mathbf{t}) > 4$. Here we have $d(\mathbf{s}) = \lambda \circ \xi_s \circ (2, 2, 4)$, where $\lambda = (4, 2)$ if $m(\mathbf{s})$ begins with 3, and $\lambda = (4, 2, 2)$ if $m(\mathbf{s})$ begins with 2. Also, we write $d(\mathbf{t}) = (4, 2, 2) \circ \xi_t \circ \mu$, as above. Then

$$\begin{aligned} d(\tilde{\mathbf{s}}) &= \lambda \circ \xi_s \circ (2, 2, 2, 6, 2, 2, 2) \circ \xi_t \circ \mu \\ &= \lambda \circ \xi_{s+t+1} \circ \mu, \end{aligned}$$

and $s + t + 1 = [l(\tilde{\mathbf{s}})/4]$ since both $l(\mathbf{s})$ and $l(\mathbf{t})$ are congruent to 2 or 3, mod 4. Thus, $\tilde{\mathbf{s}}$ is special, of type II, in any case.

Let $\mathbf{x}$ be the basic extraspecial segment of $\mathbf{h}$ involving (h_r), and in the notation of (4.2) put $\mathbf{s} = \mathbf{s}_{i-1}$, $\mathbf{t} = \mathbf{s}_i$. Then put

$$\tilde{\mathbf{x}} = \left((h_{k_1}) \circ \mathbf{s}_1 \circ \cdots \circ (h_{k_{i-1}})\right)^h \circ \tilde{\mathbf{s}} \circ (h_{k_{i+1}}) \circ \cdots \circ \mathbf{s}_{k_u} \circ (h_{k_{u+1}}).$$

It now follows that $\tilde{\mathbf{x}}$ is an extraspecial segment of $\tilde{\mathbf{h}}$ relative to $\tilde{\Sigma}$. Taking $\mathbf{x} = \mathbf{x}_k$ is the notation of (4.1), we have

$$\tilde{\mathbf{h}} = (\mathbf{a}_1 \circ \mathbf{x}_1 \circ \cdots \circ \mathbf{a}_k)^h \circ \tilde{\mathbf{x}} \circ \mathbf{a}_{k+1} \circ \cdots \circ \mathbf{x}_m \circ \mathbf{a}_{m+1},$$

and evidently $\tilde{\mathbf{h}}$ and $\tilde{\Sigma}$ are married. This completes the proof of (5.7).

5.8. THEOREM. *$\mathscr{P}$ controls convergence relative to $(b, \mathscr{D})$.*

Proof. This is simply the union of the results (5.1) through (5.7), with help from (4.9).

5.9. COROLLARY. *$\mathscr{H}$ has the $(b, \mathscr{D})$-convergence property.*

Proof. This is immediate from (5.8) and from the main result of [5].

6. THE MAIN RESULTS

Recall that we have $F = F(S)$, the free group with free basis S. For any subset T of S, identify $F(T)$ with the subgroup of F generated by T. Then let X_T denote the subtree of X induced on the vertex-set $\{g, \langle a \rangle g : g \in F(T),\ a \in T\}$. Put $N = \langle \mathscr{H} \rangle$. Recall $N_{a,b} = \langle h_{a,b}^{F(\{a,b\})} \rangle$.

6.1. THEOREM. *Let T be a subset of S, and put $N_T = \langle N_{a,b}^g : g \in F(T), a, b \in T, a \neq b \rangle$. Then $N \cap F(T) = \langle \mathscr{H} \cap F(T) \rangle = N_T$.*

Proof. Evidently $N \cap F(T) \supseteq \langle \mathscr{H} \cap F(T) \rangle \supseteq N_T$. Assume (6.1) to be false and let $x \in N \cap F(T)$ with $x \notin N_T$. Among all such x, let x be chosen so that the length $n+1$ of x as a word in the alphabet $\mathscr{H}$ is as small as possible. Let $\mathbf{h} = (h_0, \ldots, h_n)$ be such a minimal-length word for x. Then $\mathbf{h} \in \mathscr{H}^*$ by (2.0)(a), and by Corollary (5.9) we may assume that x converges to 1 via $\mathbf{h}$, relative to $(b, \mathscr{D})$. In particular, the closed interval $[x, 1] \cap A(h_0)$ has length at least $\frac{1}{2}a(h_0)$. But $a(h_0) \geq b(h_0) = 4 \cdot m(h_0) \geq 8$. This certainly implies that $[x, 1] \cap A(h)$ contains a segment I of the form $[\langle a \rangle g, \langle b \rangle g]$ for some $a, b \in S$ with $a \neq b$, and some $g \in F$. But $[x, 1] \subseteq X_T$, so $g \in F(T)$ and $a, b \in T$. Now (1.7)(d) and (1.6) together yield $h_0 \in N_{a,b}^g \leq N_T$.

Now $xh_0 \in N \cap F(T)$ and $xh_0 = (h_1 \cdots h_n)^{-1}$, so the minimality of the length of x yields $xh_0 \in N_T$, and then $x \in N_T$, a contradiction.

6.2. COROLLARY. *Let T be a subset of S, and let M_T be the Coxeter matrix $(m_{a,b})$ for $a, b \in T$. Then the canonical projection of $F(T)$ onto the Artin group $G(M_T)$ has the kernel $N \cap F(T)$. Thus, $G(M_T)$ may be identified with the subgroup $F(T)N/N$ of the Artin group G.*

Proof. The kernel of the canonical projection of $F(T)$ onto $G(M_T)$ is precisely the group N_T defined in (6.1). Then (6.1) yields a canonical isomorphism $F(T)N/N \cong G(M_T)$.

Corollary (6.2) is simply a re-statement of Theorem 2. It now remains to take up the Word Problem and Theorem 1. For the remainder of this paper, we assume that S is finite.

Put $S' = S \cup S^{-1}$, and let $FM(S')$ denote the free monoid on S'. Thus, elements of $FM(S')$ are finite sequences of elements of S', and they will be referred to as *words*. The product $w \circ w'$ of two words is obtained by the usual operation of concatenating sequences. A word $w = (x_1, \ldots, x_l)$ is *reduced* if $x_{i+1} \neq x_i^{-1}$ for any i, $1 \leq i < l$. The set of reduced words will be denoted by $\mathrm{Red}(S')$. There is a natural bijection β: $\mathrm{Red}(S') \to F$ given by $\beta(w) = x_1 \cdots x_l$.

A *section* of a word w is defined to be a triple (w_1, v, w_2) where $w = w_1 \circ v \circ w_2$.

Using β, the notions of *length* and of *syllable-length* lift from F to $\mathrm{Red}(S')$. Thus, $|w|$ is simply the length of w as a sequence, while

$$\|w\| = \|\beta(w)\| = m$$

if we can write

$$g = \beta(w) = a_{i_1}^{e_1} \cdots a_{i_m}^{e_m}$$

as in (1.2). A section (w_1, v, w_2) of w will be said to be *syllable-complete* if we have

$$\beta(w_1) = a_{i_1}^{e_1} \cdots a_{i_{j-1}}^{e_{j-1}}, \qquad \beta(v) = a_{i_j}^{e_j} \cdots a_{i_k}^{e_k}, \qquad \beta(w_2) = a_{i_{k+1}}^{e_{k+1}} \cdots a_{i_m}^{e_m}$$

for some indices j and k, with $1 \leq j \leq k \leq m$. Thus, (w_1, v, w_2) is syllable-complete if w_1 doesn't end in the same letter with which v begins, and w_2 doesn't begin with the same letter in which v ends. Let $\mathscr{S}(w)$ denote the set of all syllable-complete sections of w. For any $a, b \in S$ with $a \neq b$, put

$$\mathscr{S}_{a,b}(w) = \{(w_1, v, w_2) \in \mathscr{S}(w) : \beta(v) \in \langle a, b \rangle\},$$
$$\mathscr{S}_{a,b}^{*}(w) = \{(w_1, v, w_2) \in \mathscr{S}_{a,b}(w) : \|v\| = m_{a,b}\}.$$

We then set

$$\mathscr{S}_{a,b} = \bigcup\{\mathscr{S}_{a,b}(w) : w \in \mathrm{Red}(S')\},$$
$$\mathscr{S}^*_{a,b} = \bigcup\{\mathscr{S}^*_{a,b}(w) : w \in \mathrm{Red}(S')\}.$$

Let $G_{a,b}$ denote the Artin group with generators a and b, and with the single relation $h_{a,b} = 1$ (from (0.2)). Identify $G_{a,b}$ with $\langle a, b\rangle/N_{a,b}$, where $N_{a,b}$ is the smallest normal subgroup of $\langle a, b\rangle$ containing $h_{a,b}$.

6.3. Definition. Let $w \in \mathrm{Red}(S')$ and let $a, b \in S$ with $a \neq b$. Define three subsets of $\mathscr{S}_{a,b}$, as follows.

$\tilde{\mathscr{M}}_{a,b}(w) =$ the set of all $(w_1, u, w_2) \in \mathscr{S}_{a,b}$ such that $\|u\| \leq m_{a,b}$, and such that there exists $v \in \mathrm{Red}(S')$ with $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$ and with $\beta(u^{-1}v) \in N_{a,b}$.

$\mathscr{M}_{a,b}(w) =$ the set of all $(w_1, u, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$ such that either $|u| < |v|$, or else $|u| = |v|$ and $\|u\| < \|v\|$, where $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$.

$\mathscr{M}^*_{a,b}(w) =$ the set of all $(w_1, u, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$ such that $|u| = |v|$ and $\|u\| = \|v\| = m_{a,b}$, where $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$.

6.4. Proposition. *For any $w \in \mathrm{Red}(S')$, and for any $a, b \in S$ with $a \neq b$, the sets $\tilde{\mathscr{M}}_{a,b}(w)$, $\mathscr{M}_{a,b}(w)$, and $\mathscr{M}^*_{a,b}(w)$ are effectively constructible. Further, we have*

$$\tilde{\mathscr{M}}_{a,b}(w) = \mathscr{M}_{a,b}(w) \coprod \mathscr{M}^*_{a,b}(w) \coprod \{(w_1, v, w_2) \in \mathscr{S}_{a,b}(w) : \|v\| < m_{a,b}\}.$$

Proof. The set $\mathscr{S}(w)$ of syllable-complete sections of w is constructible in quadratic time in $|w|$. For any $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$, and any $u \in \mathrm{Red}(\{a, b, a^{-1}, b^{-1}\})$ with $\|u\| \leq m_{a,b}$, (1.4)(a) says that if $\beta^{-1}(uv) \in N_{a,b}$ then $|u| \leq |v|$. Thus, in order to construct $\tilde{\mathscr{M}}_{a,b}(w)$, we may proceed as follows. First, construct $\mathscr{S}_{a,b}(w)$ (in quadratic time). Then, for each $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$, make a list of all $u \in \mathrm{Red}(\{a, b, a^{-1}, b^{-1}\})$ with $|u| \leq |v|$, and with $\|u\| \leq m_{a,b}$. (There are on the order of at most $4^{|w|}$ such words u.) For any such u, one can decide, in polynomial time on $|v|$, whether $\beta(u^{-1}v)$ is in $N_{a,b}$. This follows from the construction of canonical forms for elements of $G_{a,b}$, as given in [3, p. 677]. Among all such n for which $\beta(u^{-1}v) \in N_{a,b}$, choose only those u such that (w_1, u, w_2) is syllable-complete. In this way we have constructed $\tilde{\mathscr{M}}_{a,b}(w)$, in exponential time on $|w|$. The subsets $\mathscr{M}_{a,b}(w)$ and $\mathscr{M}^*_{a,b}(w)$ are then easily constructed, as well.

Now let $(w_1, u, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$, and let v be the element of $\mathrm{Red}(S')$ with $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$. As observed above, we have $|u| \leq |v|$, from (1.4)(a). Suppose that $(w_1, u, w_2) \notin \mathscr{M}_{a,b}(w)$. Then $|u| = |v|$ and $m_{a,b} \geq \|u\| \geq \|v\|$.

If now $\|v\| < m_{a,b}$, then (1.4)(b) yields $u = v$, and so $(w_1, v, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$. On the other hand, if $m_{a,b} = \|u\| = \|v\|$ then $(w_1, u, w_2) \in \mathscr{M}^*_{a,b}(w)$. This proves the second part of the proposition.

We can now state an algorithm which (as will be shown) solves the Word Problem for our locally non-spherical Artin group G.

6.5. ALGORITHM $\mathscr{A}$.

(0) **Start** with a given $\tilde{w} \in \mathrm{Red}(S')$. Put $\mathscr{L} = \{\tilde{w}\}$ and $\mathscr{L}' = \emptyset$. If $\tilde{w} = (\emptyset)$ go to (YES). Otherwise, go to (1).

(1) For every $w \in \mathscr{L}$ and every $a, b \in S$ with $a \neq b$, construct the sets $\mathscr{M}_{a,b}(w)$ and $\mathscr{M}^*_{a,b}(w)$. If $\mathscr{M}_{a,b}(w) \neq \emptyset$ for some w and some a and b, go to (2). Otherwise go to (2*).

(2) Choose $w \in \mathscr{L}$ and $a, b \in S$ with $\mathscr{M}_{a,b}(w) \neq \emptyset$, and then choose $(w_1, u, w_2) \in \mathscr{M}_{a,b}(w)$. If $w_1 \circ u \circ w_2 = (\emptyset)$, go to (YES). Otherwise replace $\mathscr{L}$ by $\{w_1 \circ u \circ w_2\}$ and replace $\mathscr{L}'$ by $\emptyset$, and go to (1).

(2*) Construct the set $\mathscr{L}''$ of all $w' = w_1 \circ u \circ w_2$ such that $(w_1, u, w_2) \in \mathscr{M}^*_{a,b}(w)$ for some $w \in \mathscr{L}$ and some $a, b \in S$. If $\mathscr{L}'' \subseteq \mathscr{L} \cup \mathscr{L}'$ go to (NO). Otherwise replace $\mathscr{L}'$ with $\mathscr{L} \cup \mathscr{L}'$, and replace $\mathscr{L}$ with $\mathscr{L}'' - (\mathscr{L} \cup \mathscr{L}')$, and go to (1).

(YES) means that $\tilde{w}$ represents 1 in G.

(NO) means that $\tilde{w}$ does not represent 1 in G.

The remainder of this section will be devoted to the proof that $\mathscr{A}$ is an effective algorithm which solves the Word Problem for G. As a first step, it is important to see that $\mathscr{A}$ does indeed return an answer of (YES) or (NO) to any $\tilde{w} \in \mathrm{Red}(S')$.

Apart from the initial step from (0) to (1) (or to (YES)), and apart from the final step from (2) to (YES), or from (2*) to (NO), the algorithm $\mathscr{A}$ consists of a sequence of loops, each of which is of the form (1) $\to$ (2) $\to$ (1) or (1) $\to$ (2*) $\to$ (1). A trivial induction argument then shows that at any stage, the list $\mathscr{L}$ consists of words whose lengths are pair-wise equal. Now, in the course of a loop (1) $\to$ (2) $\to$ (1), $\mathscr{L}$ is replaced by a "new" $\mathscr{L}$ which consists of a singleton $\{w'\}$, and w' then has the property that $|w'| < |w|$ for any member w of any "old" version of $\mathscr{L}$. Thus, the "new" $\mathscr{L}$ is disjoint from the union of all the old versions of $\mathscr{L}$ in this case. On the other hand, in the course of a loop (1) $\to$ (2*) $\to$ (1), $\mathscr{L}$ is replaced by a "new" $\mathscr{L}$ which is disjoint from $\mathscr{L} \cup \mathscr{L}'$. But here $\mathscr{L} \cup \mathscr{L}'$ is the union of all the "old" versions of $\mathscr{L}$ whose members are of the minimal length achieved by the algorithm so far. Since the members of the "new" $\mathscr{L}$ also have this minimal length, it follows that, again, the "new" $\mathscr{L}$ is disjoint from the union of all the old versions of $\mathscr{L}$.

Put $l = |\tilde{w}|$. There are then at most $\mathrm{card}(S')^{(l^2+l)/2}$ reduced words of length at most l, so it now follows that after somewhat less than this number of passages through (2) and (2*), the list $\mathscr{L}$ will either be empty or will consist of the empty word. Thus, we have shown:

6.6. LEMMA. *For any $\tilde{w} \in \mathrm{Red}(S')$, the algorithm $\mathscr{A}$ returns an answer of (YES) or (NO), in exponential time relative to $|\tilde{w}|$.*

6.7. LEMMA. *Let $w \in \mathrm{Red}(S')$, put $g = \beta(w)$, and consider the oriented closed segment $[g, 1]$ from g to 1 in X. Let $\mathscr{J}(w)$ denote the set of oriented closed segments $[g_1, g_2] \leq [g, 1]$, such that g_1 and g_2 are elements of F. There is a bijection θ_w from $\mathscr{J}(w)$ to the set $\mathscr{S}(w)$ of all syllable-complete sections of w, given by*

$$\theta_w([g_1, g_2]) = \left(\beta^{-1}\left(gg_1^{-1}\right), \beta^{-1}\left(g_1 g_2^{-1}\right), \beta^{-1}(g_2)\right),$$

$$\theta_w^{-1}(w_1, v, w_2) = \left[\beta(v \circ w_2), \beta(w_2)\right].$$

Moreover, if $g_1 g_2^{-1} \in \langle a, b\rangle$, where $a, b \in S$ and $a \neq b$, then $\theta_2([g_1, g_2]) \in \mathscr{S}_{a,b}(w)$.

Proof. For any $g' \in F$ we may write

$$g' = a_{i_1}^{e_1} \cdots a_{i_k}^{e_k},$$

where $a_{i_1}, \ldots, a_{i_k} \in S$, $e_j \in \mathbb{Z}$, and $k = \|g'\|$. Put $x_0 = 1$, and define y_j and x_j, inductively, by

$$y_j = \langle a_{i_{k-j+1}}\rangle x_{j-1},$$

$$x_j = a_{i_{k-j+1}}^{e_{k-j+1}} x_{j-1},$$

for all j with $1 \leq j \leq k$. Then $g' = x_k$, and the vertex-sequence for the segment $[g, 1]$ is

$$(x_k, y_k, x_{k-1}, y_{k-1}, \ldots, x_1, y_1, x_0).$$

Now take $g' = g$, where we are given $[g_1, g_2] \leq [g, 1]$. Thus, there exist indices r and s with $1 \leq r \leq s \leq k$, such that

$$g_1 = a_{i_r}^{e_r} \cdots a_{i_1}^{e_1}, \qquad g_2 = a_{i_s}^{e_s} \cdots a_{i_1}^{e_1},$$

and it is then plain that $(\beta^{-1}(gg_1^{-1}), \beta^{-1}(g_1 g_2^{-1}), \beta^{-1}(g_2))$ is a syllable-complete section of $w = \beta^{-1}(g)$, in $\mathscr{S}_{a,b}(w)$ if $g_1 g_2^{-1} \in \langle a, b\rangle$.

On the other hand, let $(w_1, v, w_2) \in \mathscr{S}(w)$. We then have indices r and s as above, with

$$\beta(w_1) = a_{i_1}^{e_1} \cdots a_{i_{r-1}}^{e_{r-1}}, \qquad \beta(v) = a_{i_r}^{e_r} \cdots a_{i_s}^{e_s}, \qquad \beta(w_2) = a_{i_{s+1}}^{e_{s+1}} \cdots a_{i_k}^{e_k}.$$

It is then plain that $[\beta(vw_2), \beta(w_2)] \leq [g, 1]$, and that the maps that we have now obtained between $\mathscr{J}(w)$ and $\mathscr{S}(w)$ are inverse to each other.

6.8. Let $w \in \operatorname{Red}(S')$ and put $g = \beta(w)$. We will denote by $\mathscr{H}(g)$ the set of all $h \in \mathscr{H}$ such that

$$d(gh, 1) + a(h) \leq d(g, 1) + b(h),$$

$$\text{with strict inequality if } \partial(A(h) \cap [g, gh] \cap [g, 1]) \not\subseteq D. \quad (*)$$

For $h \in \mathscr{H}(g)$, let $I = I(g, h)$ denote the closed segment $A(h) \cap [g, gh] \cap [g, 1]$, oriented so that

$$I = [\partial_0 I, \partial_1 I] \leq [g, 1].$$

Notice that (0.4) and $(*)$ together yield $l(I) \geq a(h) - \frac{1}{2}b(h)$. In particular, I is non-degnerate. We then define $\bar{I} = \bar{I}(g, h)$ to be the shortest, oriented closed segment such that

$$I \leq \bar{I} \leq [g, 1], \qquad \text{with } \partial\bar{I} \subseteq D.$$

Thus, $l(\bar{I}) = l(I) + \operatorname{card}(\partial I - D) \leq l(I) + 2$.

6.9. Proposition. *Let $w \in \operatorname{Red}(S')$. Put $g = \beta(w)$, let $h \in \mathscr{H}(g)$, and put $w' = \beta^{-1}(gh)$. Then there exists $(w_1, u, w_2) \in \mathscr{M}_{a,b}(w) \cup \mathscr{M}^*_{a,b}(w)$ with $w_1 \circ u \circ w_2 = w'$.*

Proof. Let g_1 and g_2 be the initial and terminal vertices, respectively, of $\bar{I}(g, h)$, so that $\bar{I}(g, h) = [g_1, g_2] \leq [g, 1]$. Let a and b be the (uniquely determined, by (1.7)(a)) elements of S such that $h \in \mathscr{H}_{a,b}$. Then $[g_1 h, g_2] \leq [gh, 1]$. Put $I = I(g, h)$ and $\bar{I} = \bar{I}(g, h)$. Also, put $J = A(h) \cap [g, gh] \cap [gh, 1]$, oriented so that $J \leq [gh, 1]$, and put $\bar{J} = [g_1 h, g_2]$. The situation is depicted in Fig. 12, which is drawn with J non-degenerate, and with $l(\bar{I}) = l(I) + 2$.

Put $(w_1, v, w_2) = \theta_2([g_1, g_2])$, as in (6.7), so that $v = \beta^{-1}(g_1 g_2^{-1})$. Put $h' = g_2 h g_2^{-1}$. Let a and b be the uniquely determined (by (1.7)(a)) elements of S such that $h \in \mathscr{H}_{a,b}$. Then (1.7)(c) shows that $h' \in N_{a,b}$ and $A(h') \subseteq X_{a,b}$. Since the vertex g_1 is at distance 1 from a vertex of $A(h)$ of the form $\langle a \rangle x$ or $\langle b \rangle x$, for some $x \in F$, it follows that $g_1 g_2^{-1}$ is at distance 1 from a vertex of $X_{a,b}$ of the form $\langle a \rangle x g_2^{-1}$ or $\langle b \rangle x g_2^{-1}$. But then $g_1 g_2^{-1}$ is a vertex of $X_{a,b}$, which is to say that $g_1 g_2^{-1} \in \langle a, b \rangle$. As $v = \beta^{-1}(g_1 g_2^{-1})$, we then have $(w_1, v, w_2) \in \mathscr{S}_{a,b}(w)$.

Put $u = \beta^{-1}(g_1 h g_2^{-1})$. Then $\beta(u^{-1} v) = (g_2 h^{-1} g_1^{-1})(g_1 g_2^{-1}) = (h')^{-1}$, so that $\beta(u^{-1} v) \in N_{a,b}$. Now, it should be clear from Fig. 12 (which comes, after all, by (0.4)) that $[g_1 h, g_2] \leq [gh, 1]$. Then (6.7) yields $\theta_{w'}([g_1 h, g_2]) = (w_1, u, w_2) \in \mathscr{S}_{a,b}(w')$. In order to obtain $(w_1, u, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$ it now remains to show that $\|u\| \leq m_{a,b}$.

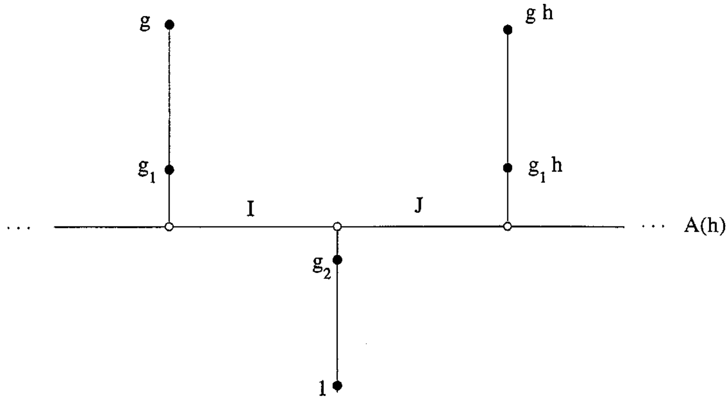


FIGURE 12

The condition (6.8)($*$) applies since $h \in \mathscr{H}(g)$, and so we have

$$a(h) - b(h) \leq l(I) - l(J).$$

Since $a(h) = l(I) + l(J)$, and since $b(h) = 4 \cdot m(h)$ by definition, we then obtain $l(J) \leq 2 \cdot m(h)$. Moreover, all of the inequalities are strict if $I \neq \bar{I}$. Notice also that if $l(\bar{J}) = l(J) + 2$ then $l(J)$ is even, and so $l(J) \leq 2 \cdot m(h) - 2$. We conclude that, in any case, we have $l(\bar{J}) \leq 2 \cdot m(h)$. Here $l(\bar{J}) = d(g_1 h, g_2) = d(\beta(u), 1)$, and so (1.3) gives $\|u\| \leq m(h)$. But $m(h) = m_{a,b}$ as $h \in \mathscr{H}_{a,b}$, so we obtain $(w_1, u, w_2) \in \tilde{\mathscr{M}}_{a,b}(w)$, as desired.

We have seen that $\beta(u^{-1}v) = (h')^{-1}$. Then $u \neq v$, as $h' \neq 1$. It now follows from the second statement in (6.4) that $(w_1, u, w_2) \in \mathscr{M}_{a,b}(w) \cup \mathscr{M}^*_{a,b}(w)$.

6.10. LEMMA. *Let* $(w_1, u, w_2) \in \mathscr{M}^*_{a,b}(w)$. *Put* $w' = w_1 \circ u \circ w_2$, $g = \beta(w)$, $g' = \beta(w')$, *and* $h = g^{-1}g'$. *Then* $h \in \mathscr{H}(g) \cup \{1\}$.

Proof. By assumption there exists $(w_1, v, w_2) \in \mathscr{S}^*_{a,b}(w)$ with $\beta(u^{-1}v) \in N_{a,b}$. If $u = v$ then $h = 1$ and we are done. So assume $u \neq v$ and put $h' = \beta(v^{-1}u)$. Then $h = g^{-1}g' = \beta(w^{-1})\beta(w') = \beta(w_2)^{-1}h'\beta(w_2)$. This shows that $h \in \mathscr{H}$. Now put $g_2 = \beta(w_2)$ and $g_1 = \beta(v \circ w_2)$. Then $[g_1, g_2] \leq [g, 1]$ by (6.7). Further, (6.7) shows that $d(g_1, g_2) = 2 \cdot \|v\|$ and $d(g_1 h, g_2) = 2 \cdot \|u\|$. Since $\|u\| = \|v\| = m_{a,b}$ we then get $d(g_1, g_1 h) \leq 4 \cdot m_{a,b}$. On the other hand, (1.12) yields $d(x, x \cdot h) \geq 4 \cdot m_{a,b}$ for any vertex x of X, with equality only if $a(h) = 4 \cdot m_{a,b}$, in which case (0.5) implies that $[x, x \cdot h]$ is a segment of $A(h)$. Thus, $[g_1, g_1 h]$ is a segment of $A(h)$ of length $a(h) = b(h)$. Now set $I = A(h) \cap [g, g'] \cap [g, 1]$. Then $[g_1, g_2] \leq I$, and thus $l(I) \geq \frac{1}{2}a(h)$. It now follows that $d(g,1) \geq d(g',1)$. Moreover, the inequality is strict unless $I = [g_1, g_2]$. This shows that $h \in \mathscr{H}(g)$, in the sense of (6.8).

6.11. We may now prove that $\mathscr{A}$ solves the Word Problem for G. Suppose false, and let $w \in \mathrm{Red}(S')$. If $\mathscr{A}$ returns an answer of (YES) to w, then it should be clear that w is indeed a word for the identity element of G. By (6.6) we may then assume that $\mathscr{A}$ says (NO) to w, and that w is a word for the identity element. Among all such w, choose w so that $l = |w| + \|w\|$ is as small as possible.

Put $g = \beta(w)$. Then $g \in \langle \mathscr{H} \rangle$, and we may choose an $\mathscr{H}$-sequence $\mathbf{h} = (h_0, \ldots, h_n)$ of minimal length for $g^{-1} = h_0 \cdots h_n$. By (2.0)(a) we then have $\mathbf{h} \in \mathscr{H}^*$ and $g^{-1} = h'_0 \cdots h'_n$ for any braiding $(h'_0, \ldots, h'_n)$ of $\mathbf{h}$. Then by Corollary (5.6) we may assume that $\mathbf{h}$ has been chosen so that the vertex g converges to 1 via $\mathbf{h}$, relative to $(b, \mathscr{D})$, in the sense of (2.5).

We now begin to follow w through $\mathscr{A}$. We have $w \neq (\emptyset)$ (else $\mathscr{A}$ immediately says (YES)), so we go to step (1), with $\mathscr{L} = \{w\}$ and $\mathscr{L}' = \emptyset$. If $\mathscr{M}_{a,b}(w) \neq \emptyset$ for some $a, b \in S$ we go to (2), replacing w by some w' with $\beta(w') \in \langle \mathscr{H} \rangle$ and with $|w'| + \|w'\| < l$. By minimality of l, $\mathscr{A}$ says (YES) to w', hence to w also, for a contradiction. We conclude that $\mathscr{M}_{a,b}(w) = \emptyset$ for all $a, b \in S$ with $a \neq b$, and we go to (2*) instead of (2).

The algorithm $\mathscr{A}$ wil now continue to loop through steps (1) and (2*), until it sends us to (NO) (i.e., until $\mathscr{L}$ becomes $\emptyset$). If at any stage we go to (2), we will reach a contradiction as above, via the minimality of l. On the other hand, for any k with $1 \leq k \leq n+1$, the kth passage through (2*) will yield $\beta^{-1}(gh_0 \cdots h_{k-1}) \in \mathscr{L}$. In order to see this, let us proceed slowly through (2*) for the kth time, $k \geq 1$. For any j with $1 \leq j \leq k$ put $x_j = \beta^{-1}(gh_0 \cdots h_{j-1})$, and put $x_0 = w = \beta^{-1}(g)$. Now form the list $\mathscr{L}''$ of all $w' = w_1 \circ u \circ w_2$ such that $(w_1, u, w_2) \in \mathscr{M}^*_{a,b}(w)$ for some $a, b \in S$. By induction on k, x_{k-1} is a member of the list $\mathscr{L}$ as we enter step (2*). Here we have $\beta(x_{k-1}) = gh_0 \cdots h_{k-2}$, and we have $h_{k-1} \in \mathscr{H}(gh_0 \cdots h_{k-2})$ in the sense of (6.8), since g converges to 1 via $\mathbf{h}$. Now (6.9) yields $x_k = w_1 \circ u \circ w_2$ for some $(w_1, u, w_2) \in \mathscr{M}^*_{a,b}(w)$, and we get $x_k \in \mathscr{L}''$.

We now claim that $x_k \notin \mathscr{L} \cup \mathscr{L}'$. (Here $\mathscr{L}$ and $\mathscr{L}'$ have their "old" values. We haven't progressed far enough through (2*), yet, to update $\mathscr{L}$ and $\mathscr{L}'$.) Suppose false. Now, any element of $\mathscr{L} \cup \mathscr{L}'$ is constructed from w in the course of $k-1$ passes through (2*). It then follows from (6.10) that there are elements $h'_1, \ldots, h'_{k-1}$ of $\mathscr{H}$ such that $x_k = \beta^{-1}(gh'_1 \cdots h'_{k-1})$. Then $\beta(x_k) = gh_0 \cdots h_{k-1} = gh'_1 \cdots h'_{k-1}$, and $g^{-1} = h'_1 \cdots h'_{k-1} h_k \cdots h_n$, contrary to the minimality of $l(\mathbf{h})$ as a word for g^{-1} in the alphabet $\mathscr{H}$. Thus $x_k \notin \mathscr{L} \cup \mathscr{L}'$, as claimed. We now finish step (2*), and arrive at a "new" $\mathscr{L}$ with $x_k \in \mathscr{L}$.

In this way we finally get $x_{n+1} \in \mathscr{L}$. But $x_{n+1} = \beta^{-1}(gh_0 \cdots h_n) = (\varnothing)$. Since $|x_0| = |x_1| = \cdots = |x_{n+1}|$, it follows that w as $(\varnothing)$ to begin with, for a contradiction. Thus, $\mathscr{A}$ solves the Word Problem for G.

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